Assignment 10

Due date: Wednesday, April 23

Hamilton’s principle with Hamiltonian variables

Hamilton’s stationary action principle is usually presented with Lagrangian variables, but it has a nice counterpart for Hamiltonian variables.

Consider the following functional for a system of \( N \) degrees of freedom:

\[
S[q_1(t), \ldots, q_N(t); p_1(t), \ldots, p_N(t)] = \int_{t_1}^{t_2} \left( \sum_{i=1}^{N} p_i \dot{q}_i - H \right) dt.
\]

You will recognize the integrand as the Lagrangian when expressed in terms of the Hamiltonian \( H \). However, you do not need this fact when solving the problem. Calculate the first order variation \( \delta S \) when

\[
q_i(t) = \tilde{q}_i(t) + \delta q_i(t) \quad i = 1, \ldots, N
\]

and

\[
p_i(t) = \tilde{p}_i(t) + \delta p_i(t) \quad i = 1, \ldots, N,
\]

and show that if it vanishes for independent variations of all \( 2N \) \( \delta q \)'s and \( \delta p \)'s, then the \( 2N \) functions \( \tilde{q} \) and \( \tilde{p} \) satisfy Hamilton’s equations. As in the Lagrangian variational principle, it will be necessary for you to assume the variations \( \delta q \) vanish at the endpoints. Is there a corresponding restriction on the variations \( \delta p \)? Is this variational principle still valid when the Hamiltonian has direct time dependence?

Preserving the form of Hamilton’s equations

Earlier in the semester you showed that arbitrary transformations of the Lagrangian variables did not change the form of the Euler-Lagrange equations (Goldstein 1.10). Is this also true for Hamilton’s equations?

Consider the most general time-independent transformation, in the case of one degree of freedom:

\[
Q = Q(q, p) \quad P = P(q, p).
\]

The original and transformed Hamiltonians are related by

\[
H(q, p) = \tilde{H}(Q, P).
\]
We would like to know, as in the Lagrangian case, whether the equations

\[
\dot{Q} = \frac{\partial \tilde{H}}{\partial P}, \quad \dot{P} = -\frac{\partial \tilde{H}}{\partial Q} \tag{1}
\]

are equivalent to

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{2}
\]

for an arbitrary transformation.

Start by expressing \(\dot{Q}\) and \(\dot{P}\) in (1) in terms of \(\dot{q}\) and \(\dot{p}\) using the chain rule, and also use the chain rule to express partial derivatives of \(\tilde{H}\) in terms of partial derivatives of \(H\). You will be able to eliminate the \(\dot{p}\) term by multiplying one equation by \(\partial P/\partial p\), the other by \(\partial Q/\partial p\), and subtracting. The resulting equation will look like the first equation in (2) apart from three factors:

\[
\frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \quad \frac{\partial q}{\partial Q} \frac{\partial P}{\partial p} + \frac{\partial p}{\partial Q} \frac{\partial Q}{\partial p}
\]

State why (cite properties of partial derivatives) one of these is identically 0, another is identically 1, and the remaining one is the Jacobian of the variable transformation. This shows that not any transformation preserves the form of Hamilton’s equations, but only those that have unit Jacobian. In mechanics, transformations with this property are called “canonical”.

You will arrive at the same conclusion when eliminating \(\dot{q}\) to get the \(\dot{p}\) equation, but you don’t have to provide those details.

**Time evolution is a canonical transformation**

Show that the Hamiltonian variable transformation (one degree of freedom) defined by time evolution,

\[
Q(q(0), p(0)) = q(\Delta t) \\
P(q(0), p(0)) = p(\Delta t)
\]

is canonical to first order in \(\Delta t\).
Poisson bracket notation

Bracketology has a rich tradition in physics, beginning with Poisson. Given arbitrary functions \( f(q, p) \) and \( g(q, p) \), he introduced the notation

\[
\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p}.
\]

Using this notation, the statement that the transformed variables \( Q \) and \( P \) are canonical is simply \( \{Q, P\} = 1 \).

Consider an arbitrary function \( A(q, p, t) \). Using the chain rule and something else, show that

\[
\dot{A} = \{A, H\} + \frac{\partial A}{\partial t},
\]

where \( H \) is the Hamiltonian. What well-known equations does this give you when \( A = q, A = p \) or \( A = H \)? State a condition, involving the Poisson bracket, such that a function \( I(q, p) \) is conserved.

The close correspondence between Poisson bracket equations and equation in quantum mechanics featuring the commutator of matrices served as an inspiration to 20th century bracketologists such as Dirac.

LC circuit Hamiltonian

An electrical \( LC \)-circuit is a series arrangement of a solenoid of inductance \( L \) with a parallel-plate capacitor of capacitance \( C \). In the limit of low electrical resistance, the dynamics of the charge \( Q(t) \) on one of the capacitor plates is described by the Lagrangian

\[
L(Q, \dot{Q}) = \frac{1}{2} L \dot{Q}^2 - \frac{1}{2} \frac{Q^2}{C}.
\]

The current flowing in the circuit is \( i = \dot{Q} \).

On the basis of this Lagrangian, is the energy stored in the inductor “kinetic” or “potential” in nature?

Determine the momentum \( P \) conjugate to the charge \( Q \) and write down the Hamiltonian \( H(Q, P) \) associated with the \( LC \)-circuit.

What are the units of \( P \)?

Show that the product \( QP \) still has units of action.
Time-dependent canonical transformations

Are we free to define canonical transformations on the Hamiltonian variables that depend on time, or is that asking for trouble? Consider the following transformation for a single degree of freedom system that depends on time ($m$ is a parameter with units of mass):

\[ Q = \frac{p}{mt}, \quad P = -\frac{m}{t}q. \]

Verify that this transformation is canonical.

Suppose the system is a free particle with Hamiltonian

\[ H(q, p, t) = \frac{p^2}{2m}. \]

Obtain the most general solution for $p(t)$. Using the canonical transformation, what then is the most general solution for $Q(t)$?

Now write down the transformed Hamiltonian $\tilde{H}$, using the rule

\[ H(q, p, t) = \tilde{H}(Q, P, t). \]

Solve Hamilton’s equation for $\dot{Q}$ using $\tilde{H}$. Is the most general solution the same you found earlier using $H$?