Electric Field of an Oscillating Charge

To determine the flux, we need the form of the electric field at the surface integral. This can be set up in either spherical or cylindrical coordinates with the motion along the $z$ axis for symmetry reasons. I will use spherical coordinates here.

\[
\hat{E} = \frac{kq}{r^2} \frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}} \hat{r}
\]

\[
\iiint \frac{kq}{r^2} \frac{(1 - \frac{v^2}{c^2})}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}} \hat{r} \cdot \hat{r} r^2 \sin \theta d\theta d\phi
\]

\[
= kq (1 - \frac{v^2}{c^2}) \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\sin \theta d\theta}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}}
\]

\[
= 2\pi kq (1 - \frac{v^2}{c^2}) \left[ \frac{\sin \theta d\theta}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}} \right]_0^\pi
\]

From this setup, we can solve the integral $I_\theta$ either with a tool like Wolfram Alpha or by hand.

( It's good practice to do it by hand, although there's no shame in getting help )
Using a few simplifications and substitutions, we can rewrite the integral as follows.

\[
I_\theta = -\int_{\frac{1}{\beta}}^{1} \frac{Y^3 \, du}{(1 + \gamma^2 \beta^2 u^2)^{3/2}} = Y^3 \int_{-1}^{1} \frac{du}{(1 + \gamma^2 \beta^2 u^2)^{3/2}}, \quad u = \cos \Theta, \quad du = -\sin \Theta \, d\Theta
\]

From here, we'll make an additional substitution

\[
sinh w = \gamma \beta \, u \quad \text{(Recall: } \cosh^2 w - \sinh^2 w = 1)\]
\[
\cosh w \, dw = \gamma \beta \, du
\]

Also, note \( \tan w = \gamma \beta u \) would work as well.

\[
= \frac{Y^2}{\gamma \beta} \int \frac{\cosh w \, dw}{(1 + \sinh^2 w)^{3/2}} = \frac{Y^2}{\gamma \beta} \int \frac{dw}{\cosh^2 w} = \frac{Y^2}{\gamma \beta} \int \text{sech}^2 w \, dw
\]

\[
= \frac{Y^2}{\gamma \beta} \tan h w \bigg|_{u=-1}^{u=1} = \frac{Y^2}{\gamma \beta} \frac{\sinh w}{\sqrt{1 + \sinh^2 w}} \bigg|_{u=-1}^{u=1}
\]

Now, since \( \sinh w = \gamma \beta u \), we have \( \text{(Note: } 1 + \gamma^2 \beta^2 = \gamma^2) \)

\[
= \frac{Y^2}{\gamma \beta} \left[ \frac{\gamma \beta}{(1 + \gamma^2 \beta^2)^{3/2}} - \frac{-\gamma \beta}{(1 + \gamma^2 \beta^2)^{3/2}} \right] = \frac{Y^2}{\gamma} \left( \frac{2 \gamma \beta}{\gamma^2} \right) = 2Y^2
\]

Finally, plugging this into the flux, we have:

\[
\frac{2\pi \, k \, q}{\gamma^2} (2Y^2) = \frac{4\pi \, k \, q}{\gamma^2} \quad \text{as expected.}
\]
Now, for the drawing. What you see below captures some of the key qualitative aspects that we are looking for.

If I place a charge along the axis perpendicular to the motion, we see that it should also begin to oscillate from side to side, even quite far away.
Motion of a charge in a uniform electric field

We can begin from the expression for the force caused by an electric field:

\[ \vec{F} = \frac{d\vec{p}}{dt} = q \vec{E} \]

For a particle of mass \( m \) and charge \( q \) in an electric field \( \vec{E} = E \hat{i} \), we find...

\[ \frac{dp}{dt} = qE \]

(where \( p = p_x \), really)

\[ \int_{t_0}^{t} dp = \int_{t_0}^{t} dt' qE \rightarrow p(t) = qEt \]

\[ \vec{p}(t) = qEt \hat{i} \]

Now, from here we know \( \vec{p}(t) = \gamma m \vec{v} \) and \( \vec{r}(t) = \vec{r}_0 + \int_{0}^{t} \vec{v} dt \)

\[ qEt \hat{i} = \frac{m \gamma \vec{v} \hat{i}}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{qEt}{m} \]

\[ \frac{v^2}{1 - \frac{v^2}{c^2}} = \frac{q^2E^2t^2}{m^2} \]

\[ v^2 \left( 1 + \frac{q^2E^2t^2}{m^2c^2} \right) = \frac{q^2E^2t^2}{m^2} \]
\[ \vec{V}(t) = \frac{qEt}{m} \sqrt{1 + \frac{q^2E^2t^2}{m^2c^2}} \hat{c}, \quad \text{and} \quad \vec{r}_o = (0, 0, 0) \]

Then, plugging in \( \vec{V}(t) \) and \( \vec{r}_o \), we find...

\[
\vec{r}(t) = \frac{qE}{m} \int_0^t \frac{dt'}{\sqrt{1 + \frac{q^2E^2t'^2}{m^2c^2}}} = \frac{mc^2}{qE} \left[ \begin{array}{c} \frac{t'}{c} \\ 1 \\ 0 \end{array} \right]_{t' = 0}^{t' = t}
\]

\[
\vec{r}(t) = \frac{mc^2}{qE} \left( \frac{1}{\sqrt{1 + \left(\frac{qEt}{mc}\right)^2}} - 1 \right) \hat{c}
\]

\[
\vec{r}(t) = \left( \sqrt{\frac{m^2c^4}{q^2E^2} + c^2t^2} - \frac{mc^2}{qE} \right) \hat{c}
\]

This does match our initial conditions and note that as \( t \to \infty \), \( V \to c \). If you've started to develop an intuition for relativity, this should match your expectation.