Consider the components of $\hat{\nabla}$ and $\hat{E}_1 \varphi_2$

\[
\begin{align*}
(\partial_x \hat{c} + \partial_y \hat{j} + \partial_z \hat{k}) \\
(E_{ix} \varphi_2 \hat{i} + E_{iy} \varphi_2 \hat{j} + E_{iz} \varphi_2 \hat{k})
\end{align*}
\]

The first term will yield:

\[
(\partial_x E_{ix}) \varphi_2 + E_{ix}(\partial_x \varphi_2)
\]

and each subsequent term will be the same with $x$ replaced by $y$ and $z$. Thus we will have terms that look like $\varphi_2 (\hat{\nabla} \cdot \hat{E}_1)$ from the first terms, and $\hat{E}_1 (\hat{\nabla} \varphi_2)$ from the second terms. Adding these together we have

\[
\hat{\nabla} \cdot (\hat{E}_1 \varphi_2) = (\hat{\nabla} \cdot \hat{E}_1) \varphi_2 + \hat{E}_1 \cdot \hat{\nabla} \varphi_2
\]

With this we can rewrite $U_{12}$ as follows.

\[
U_{12} = -\epsilon_0 \int d^3r \left( \hat{\nabla} \cdot (\hat{E}_1 \varphi_2) - (\hat{\nabla} \cdot \hat{E}_1) \varphi_2 \right)
\]
Use the divergence theorem to prove that
\[ \int d^3r \nabla \cdot (E_1 \varphi_2) = 0. \]

With the divergence theorem we can convert the above into:
\[ \oint d\mathbf{A} \cdot (\mathbf{E}_1 \varphi_2) \]

This corresponds to an integral at \( r \rightarrow \infty \). While the surface area grows as \( r^2 \), the integrand tends to zero as \( 1/r^3 \). Thus the integral as a whole goes to zero.

You are now left with evaluating the integral
\[ \int d^3r (\nabla \cdot E_1) \varphi_2. \]

Now, we can utilize Gauss’ law and recognize that the divergence of \( \mathbf{E}_1 \) can be replaced with the charge distribution associated with \( \mathbf{E}_1 \).

\[ \nabla \cdot \mathbf{E}_1 = \frac{q_1}{\varepsilon_0} \delta^3(\mathbf{r}-\mathbf{r}_1). \]

(Note: We can ignore the other charge for the same reason that we are expressing the electric field as the sum of the field created by \( q_1 \) and \( q_2 \).)
Now this integral becomes:

\[-\int d^3r \, q \, S(\vec{r} - \vec{r}_1) \, \frac{k q_2}{|\vec{r} - \vec{r}_2|} = \frac{k q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}\]

This should be familiar!

---

*Point charge near the surface of a conductor*

Keeping in mind the rules that apply to field lines near a conductor, make a drawing of the field lines in the \(x-y\) plane (the 3D pattern is symmetric about the \(x\)-axis). If this reminds you of (half of) a field line drawing you’ve seen before, state which it was.
This should be very reminiscent of the pattern shown by a dipole (or at least half of the field).

An electric field that has all the required properties in the region $x > 0$ can be constructed by placing a “fictitious” charge $-q$ on the $x$-axis at $x = -a$, and letting it together with the actual $+q$ charge at $x = +a$ be sources of the field (we ignore what is going on in the region $x < 0$ for now). Using symmetry, show that the net electric field right at the surface of the conductor, $x = 0$, is perpendicular to the surface. Compute the magnitude of this surface field, $E(r)$, as a function of the distance $r$ from the $x$-axis. As a check, your answer should have the form

$$E(r) = \frac{A}{(r^2 + B)^{3/2}},$$

where $A$ and $B$ are constants (to be determined by you).

Now if we add an additional charge to create the appropriate field, our field must respect the cylindrical symmetry of our charge distribution. Therefore, it can only be either purely perpendicular or purely parallel to the conducting plane. Since the charges are opposite signed, we expect it will be purely ___.

This makes calculating the magnitude easier since we only need to worry about the $\hat{\imath}$ component.

$$E_x(r) = kq \left[ \frac{(x-a)}{(x-a)^2 + y^2 + z^2}^{3/2} - \frac{(x+a)}{(x+a)^2 + y^2 + z^2}^{3/2} \right]$$

$$= \frac{-2kqa}{(a^2 + y^2 + z^2)^{3/2}} = \frac{-2kqa}{(a^2 + r^2)^{3/2}} = E(r)$$
Now suppose the electric field really is zero inside the conductor, \( x < 0 \). In order for it to abruptly jump to the non-zero value you found above at \( x = 0 \), there must be some surface charge density \( \sigma(r) \). Calculate \( \sigma(r) \), using Gauss’s law, as we did in lecture. Integrate this surface charge density over the entire plane \( x = 0 \) to find out how much total charge resides on the surface of the conductor.

If we set a Gaussian surface very close to the conductor, we can use the limiting case to find the charge density.

\[
\varepsilon_0 \left( \text{flux} \right) = \sigma(r) \cdot dA
\]

\[
\varepsilon_0 \left( \mathbf{E} - \mathbf{E}_0 \right) \cdot dA = \sigma(r) \cdot dA
\]

\[
\varepsilon_0 \left( \frac{-2kq a}{(a^2 + r^2)^{3/2}} \right) = \sigma(r)
\]

Now if we evaluate the integral we find:

\[
\int dA \sigma(r) = -2k\varepsilon_0 q a \int \frac{r \, dr \, d\theta}{(a^2 + r^2)^{3/2}} = -q
\]

Finally, it is easiest to evaluate the force on \( q \) due to the fictitious charge a distance \( 2a \) away.

\[
\mathbf{F_q} = -\frac{k q^2}{4a^2} \mathbf{\hat{j}}
\]