Magnetic Dipoles

Let's restrict our loop to the xy-plane with the realization that we can always rotate a loop into the xy-plane.

\[ \oint r \cdot \mathbf{V} \, dr = \oint (\hat{r} \cdot \mathbf{V}) \, (dx \, \hat{i} + dy \, \hat{j}) \]

Green's Theorem (or Stoke's Theorem, take your pick) tells us:

\[ \oint (A \, dx + B \, dy) = \iint \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, dx \, dy \]

\[ \oint r \cdot \mathbf{V} \, dr = \iint - \partial_y \left( xV_x + yV_y \right) \, dx \, dy \, \hat{i} + \partial_x \left( xV_x + yV_y \right) \, dx \, dy \, \hat{j} \]

\[ = (-V_y \, \hat{i} + V_x \, \hat{j}) \iint dx \, dy = (\text{Area}) \hat{\mathbf{z}} \times \mathbf{V} = \hat{\mathbf{a}} \times \mathbf{V} / \text{Area} = \hat{\mathbf{a}}. \]

You could also work this integral out for a small differential square and argue that any larger areas are made up of similar squares that will cancel on all sides except for the outer boundary.
In this next part you are interested in the forces on the current loop in the presence of a uniform external magnetic field $\mathbf{B}$. Starting from the force

$$dF = Idr \times \mathbf{B}$$

on an element $dr$ of the loop, first show that the net magnetic force vanishes. Next find the torque on the loop by calculating the line integral

$$T = \oint_C \mathbf{r} \times d\mathbf{F}.$$

$$\mathbf{F}_{\text{net}} = \oint C \mathbf{F} = \oint I dr \times \mathbf{B} \quad \text{(with our loop still in the xy-plane)}$$

$$= I \oint [(B_y dx - B_x dy) \mathbf{\hat{z}} + (B_z dy \mathbf{\hat{z}}) + (-B_x dx \mathbf{\hat{j}})]$$

We can apply Stokes’ theorem once again to these three integrals and convert these to:

\[
\hat{i}: \quad \iint dy \, dz \left( -\partial_z (B_y) \right) = 0
\]

\[
\hat{j}: \quad \iint dx \, dz \left( +\partial_z B_z \right) = 0
\]

\[
\hat{k}: \quad \iint dx \, dy \left( -\partial_x B_x - \partial_y B_y \right) = 0
\]

All of these are zero because $\mathbf{B}$ is uniform. Now, as for the torque,

$$T = \oint C \mathbf{r} \times (I dr \times \mathbf{B})$$

Now using the vector identity $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B})$, we can rewrite this integral.
\[ \oint_c \vec{r} \times (I \, d\vec{r} \times \hat{B}) = \oint C I \, d\vec{r}' (\hat{r} \cdot \hat{B}) - \hat{B} (\hat{r} \cdot d\vec{r}) \]

The second term will be zero, as \( \hat{r} \cdot d\vec{r} \) will always average to zero over a closed loop integral. (You can quite quickly check this with Stoke's theorem.)

The first term can be rewritten using what we learned in the first part as:

\[ I \, \hat{a} \times \hat{B} = \hat{m} \times \hat{B} \]

with \( \hat{m} = I \, \hat{a} \)

In this last part there is no external magnetic field because you are interested in the loop as a source of magnetic field. Start with the formula for the vector potential \( \vec{A} \) we derived in lecture:

\[ \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \oint C I \, d\vec{r}' \left( \frac{1}{r} \hat{r} - \frac{\hat{r} \cdot \vec{r}'}{r^2} \right) \]

The magnetic field only has the dipole form when the field point \( \vec{r} \) is far from the loop. You should therefore use just the first two terms in the expansion

\[ \frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{\hat{r} \cdot \vec{r}'}{r^2} \ldots \]

\[ \hat{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int d\vec{r}' \left( \frac{\hat{r} \cdot \vec{r}'}{r^2} \right) = \frac{\mu_0 I}{4\pi} \hat{a} \times \frac{\hat{r}}{r^2} = \frac{\mu_0}{4\pi r^2} \hat{m} \times \hat{r} \]

\[ \hat{A}(\vec{r}) = \frac{\mu_0}{4\pi r^2} \hat{m} \times \hat{r} \]
Hall effect and mobility

Suppose you are designing a Hall probe for measuring magnetic fields and want it to have the greatest sensitivity. What material should you choose?

There are two relevant voltages in a Hall effect experiment: the longitudinal voltage \( V \) that drives the current, and the transverse Hall voltage \( V_H \). Re-express the formula for \( V_H \) derived in lecture in terms of \( V \) rather than the current \( I \). Also, instead of the resistance use the material’s intrinsic properties. Your new formula will have \( V_H/V \) proportional to \( B \) with a constant that involves geometrical parameters and a single materials property, the mobility \( \mu \).

For greatest sensitivity with fixed longitudinal voltage and geometry you want the mobility to be as large as possible. For devices in the shape of a cube, calculate \( V_H/V \) when \( B = 1 \) G and two materials: copper, with \( \mu = 0.004 \), and silicon, with \( \mu = 0.15 \) (MKS units).

Recall from lecture that for the following setup:

\[
\begin{align*}
\text{I} & \rightarrow \quad \begin{array}{c}
\begin{array}{c}
\text{B} \\
\end{array}
\end{array}
\end{align*}
\]

\[
V_d = \mu I E = \mu \frac{V}{L}
\]

\[
V = \mu v_d B = \mu \frac{V}{L} B
\]

\[
\frac{V_H}{V} = \mu B \quad \text{(for a cube)}
\]

Now for \( B = 1 \) G = \( 10^4 \) T, we have

**Copper:** \( \frac{V_H}{V} = 4 \times 10^{-3} \times 10^{-4} = 4 \times 10^{-7} \)

**Silicon:** \( \frac{V_H}{V} = 1.5 \times 10^{-1} \times 10^{-4} = 1.5 \times 10^{-5} \)