Physics 6561 Fall 2017
Problem Set 9 Solutions

Amir Tajdini
at734@cornell.edu

Purcell's iconic radiation sketch

In this problem you will recreate a numerically accurate rendition of the sketch in Edward Purcell's famous textbook showing how the continuity of electric field lines, produced when a charge transitions between two states of uniform motion, explains the transverse nature and magnitude enhancement — over static fields — of radiation fields.

As in Purcell's model, the charge moves along a straight line, which for us defines the x-axis. For the position we take the convenient function

$$x(t) = v \log\left(1 + e^t\right),$$

where t is the time in our frame of reference and v is the asymptotic velocity at large positive t. Since the asymptotic velocity at large negative t is zero, this describes a transition between two states of uniform motion. The approach to these asymptotic states is exponentially fast. When properly executed our electric field plot should therefore display, as above, the simple electric fields of the uniform states of motion.

Make a plot of the electric field in the x-y plane at time $t = 40$ and in the range $-50 < x < 50$, $-50 < y < 50$. Set $v = 1/2$, so that at time $t = 40$ the particle will be at $x = 20$. All your numerical computations will be based on the formula for $\mathbf{E}$ you derived in the last assignment. First establish that the direction of $\mathbf{E}$ also lies in the x-y plane. For the most part you only need to carefully transcribe the symbols in the formula into statements of computer code (I used *Mathematica*). However, one part of the computation — determining the retarded time of the source — will probably be a new challenge. I suggest you define a function $\mathsf{tret}[t, x, y]$ that returns the retarded time at any position (x, y) and time t (your plot will be for $t = 40$). If you do not have access to a package that provides for simple root-finding you can code this by hand using bisection. In any case, be careful that your function returns the retarded-time solution rather than the advanced-time solution! To speed up your code make only one call to $\mathsf{tret}[t, x, y]$ at each (x, y) and use it in all the quantities $\boldsymbol{\beta}$, $\hat{\mathbf{n}}$, etc., that require it.

See the Mathematica file attached in the website!
Thanks to Daniel Longenecker!

A mysterious vector density

The midterm exam featured the following scalar (rotational invariant) constructed from the 3-vector potential:

$$V^0 = \mathbf{A} \cdot \nabla \times \mathbf{A}.$$

1. As the notation suggests, V^0 is the time-component of a 4-vector V^α. Write a 4-vector definition of V^α in terms of some combination of (4-vectors/tensors) A, F and $\tilde{F}$ that reduces to V^0 for $\alpha = 0$.

We know both $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ have one A and one derivative. Since V^0 is constructed from A and one derivative of A, the 4-vector V^α may be written as linear combination of following structures,

$$V^\alpha = cF^{\alpha\beta}A_\beta + \tilde{c}\tilde{F}^{\alpha\beta}A_\beta \tag{1.1}$$

where $c, \tilde{c}$ are two real numbers.
By evaluating V^0,

$$V^0 = cF^{0i}A_i + \tilde{c}\tilde{F}^{0i}A_i = cE^iA_i + \tilde{c}B^iA_i = cE^iA_i + \tilde{c}B^iA_i \tag{1.2}$$

where I used $\tilde{F}$ can be obtained from F by sending $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$.

Note that expanding we find

$$V^0 = A_iB_i \tag{1.3}$$

where sum over i as usual is implicit.
So in order to match them we need to set $c = 0$ and $\tilde{c} = -1$ ($A^iB_i = -A_iB_i$).
That leads to

$$V^\alpha = -F^{\alpha\beta}A_\beta = \tilde{F}^{\beta\alpha}A_\beta \tag{1.4}$$

2. Show that V^α has vanishing 4-divergence for the most general electromagnetic field produced by a single point charge. *Hint*: Consult the previous assignment on the value of $\mathbf{E} \cdot \mathbf{B}$.

   From this property we know that

$$M = \int d^3x\, V^0$$

is constant in time (subject to the usual boundary assumptions) — a conserved quantity! In the midterm you showed that M is gauge invariant (again, up to boundary behavior assumptions).

Now that we have covariant form of the vector, we can easily compute its divergence,

$$
\begin{aligned}
\partial_\alpha V^\alpha &= \partial_\alpha \tilde{F}^{\beta\alpha} A_\beta + \tilde{F}^{\beta\alpha}\partial_\alpha A_\beta \\
&= 0 + \frac{1}{2}\tilde{F}^{\beta\alpha}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = -\frac{1}{2}F_{\alpha\beta}\tilde{F}^{\alpha\beta} = 2\vec{E}\cdot\vec{B} \qquad (1.5)
\end{aligned}
$$

where I used Maxwell's constraints equations $\partial_\alpha \tilde{F}^{\alpha\beta} = 0$ (which is true in the absence of magnetic monopoles) and the result from homework 7.
We found the most general form of electric and magnetic field in the previous homework written as:

$$
\begin{aligned}
\vec{E} &= \frac{q}{y\cdot u}\left(\frac{\vec{R}a^0 - R\vec{a}}{y\cdot u} + \frac{\vec{R}u^0 - R\vec{u}}{(y\cdot u)^2}(1 - y\cdot a)\right) \\
\vec{B} &= \hat{n}\times\vec{E} \qquad (1.6)
\end{aligned}
$$

As it can be seen from this expression $\vec{E}\cdot\vec{B} = \vec{E}\cdot(\hat{n}\times\vec{E}) = 0$, and this shows V^α is conserved vector for a moving charged particle and M is the conserved charge associated with this current.

3. Take up the following plan of action to determine what M might be, and what scale factor should be used in its definition.

M has the form $A\partial A$. A is the quantity appearing in Lagrangian of E&M that we would extremize over different paths, so by analogy with classical physics, A should be thought as $x(t)$. With that analogy, M can be thought as $\sim x\frac{dx}{dt}$ and the quantity we know in classical mechanics having this form is angular momentum, so naive expectation tells us M should have something to do with angular momentum of field theory!

(a) Construct the 3-vector potentials $\mathbf{A}_\pm$ for pure modes of right and left circularly polarized light. You will need to multiply your modes by a very slowly varying scalar "envelope-function" to normalize them (this function is arbitrary in all other respects).

Let's work in Coloumb gauge. Without loss of generality, let's assume the light is propagating in the z direction. The right moving and left moving polarized light in this gauge can be written as

$$
\begin{aligned}
\mathbf{A}_+ &= A_+^0 \ \mathrm{Re}[\boldsymbol{\varepsilon}_+ e^{i(\omega t - kz)}] = 1/2A_+^0\left(\boldsymbol{\varepsilon}_+ e^{i(\omega t-kz)} + \boldsymbol{\varepsilon}_+^* e^{-i(\omega t-kz)}\right) = \\
\mathbf{A}_+ &= A_+^0\left(\hat{x}\cos(\omega t - kz) - \hat{y}\sin(\omega t - kz)\right) \qquad (1.7) \\
\mathbf{A}_- &= A_-^0 \ \mathrm{Re}[\boldsymbol{\varepsilon}_- e^{i(\omega t - kz)}] = 1/2A_-^0\left(\boldsymbol{\varepsilon}_- e^{i(\omega t-kz)} + \boldsymbol{\varepsilon}_-^* e^{-i(\omega t-kz)}\right) = \\
\mathbf{A}_- &= A_-^0\left(\hat{x}\cos(\omega t - kz) + \hat{y}\sin(\omega t - kz)\right) \qquad (1.8)
\end{aligned}
$$

where $\boldsymbol{\varepsilon}_+ = \mathbf{x} + i\mathbf{y}$ and $\boldsymbol{\varepsilon}_+ = \mathbf{x} - i\mathbf{y}$. Normalization of modes at the end will be the same and is related to choice of slowly varying function mentioned in question but I labeled them differently at this point.
This equations tell us polarization is transverse and perpendicular to direction of propogation which is $\hat{\mathbf{z}}$.
Also, in deriving above equations I used the fact $\nabla \cdot \mathbf{A} = 0$ which is Coloumb gauge condition. It turns out when this combined with source free Maxwell's equations we can use residual gauge symmetry to set potential $\phi(t, \mathbf{x}) = 0$ as well.
This expression should be familiar from undergraduate E&M course. For more details, look for example at chapter 9 of "Griffiths, Introduction to electrodynamics".

(b) Calculate the total energies $\mathcal{E}$ of your modes (by integrating θ^{00}) and fix the normalization by the property that $\mathcal{E} = \hbar\omega$, where ω is the mode frequency.

The total energy of modes is obtained by integrating energy density for each modes over the whole space-time,

$$\mathcal{E} = \frac{1}{8\pi} \int d^3x (\vec{E}^2 + \vec{B}^2) \tag{1.9}$$

The electric field and magnetic field can be obtained from vector potential:

$$\begin{aligned} \vec{E}_{\pm} &= -\frac{\partial \vec{A}_{\pm}}{\partial t} = A^0_{\pm}\omega(-\hat{x}\sin(\omega t - kz) \mp \cos(\omega t - kz)\hat{y}) \\ \vec{B}_{\pm} &= \nabla \times A = A^0_{\pm}k(\hat{y}\sin(\omega t - kz) \pm \cos(\omega t - kz)\hat{x}) \end{aligned} \tag{1.10}$$

Note that from dispersion relation we know $k = \omega/c = \omega$ since we set $c = 1$ from beginning.
Computing energy density we find:

$$\begin{aligned} &\vec{E}^2_{\pm} + \vec{B}^2_{\pm} = 2(A^0_{\pm})^2\omega^2 \\ &\mathcal{E}_{\pm} = \frac{2\omega^2(A^0_{\pm})^2}{8\pi} \int dxdydz f(x,y,z) = \frac{(A^0_{\pm})^2\omega^2}{4\pi} \int dxdydz f(x,y,z) = \frac{(A^0_{\pm})^2\omega^2}{4\pi} \end{aligned} \tag{1.11}$$

where f can be thought as a Gaussian function with integral equal to one or any slowly varying function with integral equal to one needed for normalizing modes.
Comparing to $\mathcal{E} = \hbar\omega$, we find:

$$A_{\pm}^{0} = \sqrt{\frac{4\pi\hbar}{\omega}} \tag{1.12}$$

(c) Keeping the same normalization, calculate the M-values of your modes. Use these results to motivate a choice of scale for the mystery property M.

We should plug in same vector potential into expression we had for M,

$$V_{\pm}^{0} = \vec{A} \cdot (\nabla \times \vec{A}) = \vec{A}_{\pm} \cdot \vec{B}_{\pm} = k(A_{\pm}^{0})^2 \tag{1.13}$$

So integrating them and multiplying them by slowly varying function yields

$$M_{\pm} = \pm\omega(A_{\pm}^{0})^2 \int d^3x f = \pm\omega\frac{4\pi\hbar}{\omega} = \pm 4\pi\hbar \tag{1.14}$$

So as we predicted, M indeed has the unit of angular momentum $\hbar$ and its measuring the spin of the wave.