Physics 6561 Fall 2017
Problem Set 8 Solutions

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Charged particle motion from the action principle

In this problem we treat the electromagnetic field as given (produced by an extrenal agent) and study its effect on the motion of a relativistic particle of mass m and charge q, The converse, or electromagnetic field produced by a given charged-particle world line, gets equal treatment in the second problem.

So as to not confuse a general event x in space-time with the particle world-line, we use the notation $\zeta(t)$ for the latter, where t is an arbitrary parameter (not necessarily time). Consider the following definition of the 4-current density in space-time associated with the particle:

$$J^\alpha(x) = q \int (\mathcal{L}dt) u^\alpha(t) \delta^4(x - \zeta(t)), \tag{1.1}$$

where

$$d\tau = \mathcal{L}dt = \sqrt{\dot{\zeta}^\alpha \dot{\zeta}_\alpha} dt \tag{1.2}$$

is the proper-time element and $u^\alpha = \dot{\zeta}^\alpha / \mathcal{L}$ is the 4-velocity.

1. Confirm that this geometrical definition mathces the usual definition of the 4-current density by choosing time $t = x^0$ as the arbitrary parameter and explicitly evaluating the integral. Specifically, show that

$$\begin{aligned} J^0(t, \mathbf{x}) &= \rho(t, \mathbf{x}) = q\delta^3(\mathbf{x} - \zeta(t)) \\ \mathbf{J}(t, \mathbf{x}) &= q\mathbf{v}(t)\delta^3(\mathbf{x} - \zeta(t)). \end{aligned} \tag{1.3}$$

Worldline given by $\zeta(t)$, t is some parameter. By choosing this parameter to be $t = \zeta^0$,

$$\begin{aligned} d\tau &= \sqrt{(\dot{\zeta}^0)^2 - \dot{\zeta}^i \dot{\zeta}^i} dt = \sqrt{1 - (\dot{\vec{\zeta}})^2} dt \\ \dot{\zeta}^0 &= d\zeta^0/dt = 1 \qquad \dot{\vec{\zeta}} = \vec{v} \end{aligned} \tag{1.4}$$

$$J^0(t,\mathbf{x}) = q\int dt(\mathcal{L}u^0)dt\delta(x^0-t)\delta^3(\mathbf{x}-\vec{\zeta}(t)) =$$
$$q\int dt(t-x^0)\delta^3(\mathbf{x}-\vec{\zeta}(t)) = q\delta^3(\mathbf{x}-\vec{\zeta}(t)) \tag{1.5}$$

$$\vec{J} = q\int dt(\mathcal{L}\vec{u})dt\delta(x^0-t)\delta^3(\mathbf{x}-\vec{\zeta}(t))$$
$$q\int \vec{v}\delta(x^0-t)\delta^3(\mathbf{x}-\vec{\zeta}(t)) = q\vec{v}\delta^3(\mathbf{x}-\vec{\zeta}(t)) \tag{1.6}$$

2. Return to the integral expression for $J^\alpha(x)$ without the specialization t = time. Show that

$$S_{\text{int}}[\zeta] = \int d^4x J^\alpha(x)A_\alpha(x) = q\int dt\dot{\zeta}^\alpha(t)A_\alpha(\zeta(t)), \tag{1.7}$$

Here S_{int} is the term in the action that couples the particle to the electromagnetic field.

By direct substitution of 4-current, we find

$$S_{\text{int}}[\zeta] = q\int d^4x\int(\mathcal{L}dt)u^\alpha(t)\delta^4(x-\zeta(t))A_\alpha(x)$$
$$= q\int dt(\dot{\zeta}^\alpha)\int d^4x\delta^4(x-\zeta(t))A_\alpha(x)A_\alpha(x) = q\int dt\dot{\zeta}^\alpha(t)A_\alpha(\zeta(t)) \tag{1.8}$$

3. Obtain the equations of motion for the particle from the combined action,

$$S[\zeta] = S_{\text{free}}[\zeta] + S_{\text{int}}[\zeta], \tag{1.9}$$

where

$$S_{\text{free}}[\zeta] = m\int \mathcal{L}dt \tag{1.10}$$

is the action of the free particle (derived in lecture) with a particular choice of scale factor. Specifically, show that the Euler- Lagrange equations imply

$$m\dot{u}^\alpha(t) = qF^{\alpha\beta}(\zeta(t))\dot{\zeta}_\beta(t) \tag{1.11}$$

Euler-Lagrange equation is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\zeta}_\alpha}\right) = \frac{\partial L}{\partial \zeta_\alpha} \tag{1.12}$$

where $L = m\sqrt{\dot{\zeta}^\beta \dot{\zeta}_\beta} + q\dot{\zeta}^\beta(t) A_\beta(\zeta(t))$. Computing Euler-Lagrange equation for this Lagrangian L, we find

$$\begin{aligned}
&\frac{\partial L}{\partial \dot{\zeta}_\alpha} = \frac{m\dot{\zeta}^\alpha}{\sqrt{\dot{\zeta}^\beta \dot{\zeta}_\beta}} + qA^\alpha \\
&\frac{\partial L}{\partial \zeta_\alpha} = q\dot{\zeta}^\beta \partial^\alpha A_\beta \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\zeta}_\alpha}\right) = m\dot{u}^\alpha + q\zeta^\beta \partial_\beta A^\alpha(\zeta) \\
&\Rightarrow \quad m\dot{u}^\alpha = q\zeta^\beta(\partial^\alpha A_\beta - \partial_\beta A^\alpha) = qF^\alpha_\beta \dot{\zeta}^\beta = qF^{\alpha\beta}\dot{\zeta}_\beta
\end{aligned} \tag{1.13}$$

4. Again using time as the parameter, express the four components of the equations of motion 1.11 in terms of **E** and **B**, and the compoents

$$\begin{aligned}
mu^\alpha &= (\mathcal{E}, \mathbf{p}) \\
\dot{\zeta}^\alpha &= (1, \mathbf{v})
\end{aligned} \tag{1.14}$$

Setting $\alpha = 0$,

$$\dot{\mathcal{E}} = qF^{0\beta}\dot{\zeta}_\beta = qF^{0j}\dot{\zeta}_j = q(-\mathbf{E}) \cdot (-\mathbf{v}) = q\mathbf{E} \cdot \mathbf{v} \tag{1.15}$$

For spatial components,

$$\begin{aligned}
&\dot{p}^i = qF^{i\beta}\zeta_\beta = qF^{ij}\dot{\zeta}_j + qF^{i0}\dot{\zeta}_0 = qE^i + q\epsilon^{ijk}B_k\dot{\zeta}_j \\
&\ = qE^i + q(\mathbf{v} \times \mathbf{B})^i \\
&\dot{\mathbf{p}} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})
\end{aligned} \tag{1.16}$$

Field produced by a relativistic charged particle

Consider a particle of charge q described by a general world-line $\zeta(\tau)$, parametrized by the proper time τ elapsed along the world-line. In lecture we derive the formula

$$\partial^\alpha A^\beta(x) = \left(\frac{q}{y \cdot u}\right) \frac{d}{d\tau}\left(\frac{y^\alpha u^\beta}{y \cdot u}\right)\bigg|_{\tau=\tau_0} \tag{2.17}$$

for the gradient of the 4-vector potential produced by the particle. As in the lecture, $u(\tau)$ is the particle 4-velocity and $y(\tau) = x - \zeta(\tau)$is the 4-vector separation between x and events along the world-line. At the parameter value $\tau = \tau_0$, the separation y is null and the source $\zeta(\tau_0)$ lies on the past light-cone of x.

1. From the above formula for $\partial^\alpha A^\beta$ obtain $F^\alpha\beta$ and show that

$$\begin{aligned}\mathbf{E} &= \frac{q}{y \cdot u}\left(\frac{\mathbf{R}a^0 - R\mathbf{a}}{y \cdot u} + \frac{\mathbf{R}u^0 - R\mathbf{u}}{(y \cdot u)^2}(1 - y \cdot a)\right) \\ \mathbf{B} &= \hat{\mathbf{n}} \times \mathbf{E}\end{aligned} \tag{2.18}$$

Here $(a^0, \mathbf{a}) = du^\alpha/d\tau$ are the components of the 4-acceleration and the null separation 4-vector is expressed in terms of a distance R and unit 3-vector as $y^\alpha = (R, \mathbf{R}) = (R, R\hat{\mathbf{n}})$.

Solution due to Joseph Mittelstaedt edited by Amir

We'll first expand out the derivative term in $\partial^\alpha A^\beta$:

$$\frac{d}{d\tau}\left(\frac{y^\alpha u^\beta}{y \cdot u}\right) = \frac{\dot{y}^\alpha u^\beta + y^\alpha \cdot u^\beta}{y \cdot u} + \frac{y^\alpha u^\beta}{(y \cdot u)^2}(\dot{y} \cdot u + y \cdot \dot{u}) \tag{2.19}$$

where everything is evaluated at $\tau = \tau_0$. We now note that $\dot{y} = -\dot{\zeta}$, and since we are parametrizing by τ then $\dot{\zeta} = u$ and $u \cdot u = 1$. Using this, we can simplify this term to

$$\frac{d}{d\tau}\left(\frac{y^\alpha u^\beta}{y \cdot u}\right) = \frac{y^\alpha a^\beta - u^\alpha u^\beta}{y \cdot u} - \frac{y^\alpha u^\beta}{(y \cdot u)^2}(1 - y \cdot a) \tag{2.20}$$

We can now find the electromagnetic field tensor as

$$\begin{aligned}F^{\alpha\beta} &= \partial^\alpha A^\beta - \partial^\beta A^\alpha \\ &= \frac{q}{y \cdot u}\left(\frac{y^\alpha a^\beta - y^\beta a^\alpha}{y \cdot u} + \frac{y^\alpha u^\beta - u^\alpha y^\beta}{(y \cdot u)^2}(1 - y \cdot a)\right)\end{aligned} \tag{2.21}$$

We then have that

$$E^i = F^{i0} = \frac{q}{y \cdot u}\left(\frac{y^i a^0 - a^i y^0}{y \cdot u} + \frac{y^i u^0 - y^0 u^i}{(y \cdot u)^2}(1 - y \cdot a)\right) \tag{2.22}$$

So that

$$\mathbf{E} = \frac{q}{y \cdot u}\left(\frac{\mathbf{R}a^0 - R\mathbf{a}}{y \cdot u} + \frac{\mathbf{R}u^0 - R\mathbf{u}}{(y \cdot u)^2}(1 - y \cdot a)\right) \tag{2.23}$$

as desired.
For magnetic field, we can repeat the same steps,

$$F^{ij} = \frac{q}{y \cdot u}\left(\frac{y^i a^j - y^j a^i}{y \cdot u} + \frac{y^i u^j - y^j u^i}{(y \cdot u)^2}(1 - y \cdot a)\right) \tag{2.24}$$

.

Comparing this with $\hat{\mathbf{n}} \times \mathbf{E}$ and using the fact $\hat{\mathbf{n}} = \mathbf{R}/R$, we find

$$\begin{aligned}
\hat{\mathbf{n}} \times \mathbf{E} &= \frac{q}{y \cdot u}\left(\frac{-\mathbf{R} \times \mathbf{a}}{y \cdot u} + \frac{-\mathbf{R} \times \mathbf{u}}{(y \cdot u)^2}(1 - y \cdot a)\right) \\
&= \frac{q\epsilon^k{}_{ij}}{2y \cdot u}\left(\frac{-(y^i a^j - y^j a^i)}{y \cdot u} + \frac{-(y^i u^j - y^j u^i)}{(y \cdot u)^2}(1 - y \cdot a)\right) \\
&= -\frac{1}{2}\epsilon^k{}_{ij}F^{ij} = B^k
\end{aligned} \tag{2.25}$$

as we wanted to show.

2. Rewrite the expression for $\mathbf{E}$ in terms of the ordinary 3-velocity β, the 3-acceleration $\dot{\beta} = d\beta/dt, \gamma = 1/\sqrt{1-\beta^2}$ and arrive at:

$$\mathbf{E} = \frac{q}{(\gamma R)^2}\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{q}{R}\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}}{(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} \tag{2.26}$$

We can get one term from writing

$$\begin{aligned}
&\frac{q}{(y \cdot u)^3}(\mathbf{R}u^0 - R\mathbf{u}) = \frac{qR\gamma}{(\gamma R)^3(1 - \hat{n} \cdot \beta)^3}(\hat{n} - \vec{\beta}) \\
&\frac{1}{(\gamma R)^2}\frac{\hat{n} - \vec{\beta}}{(1 - \hat{n} \cdot \vec{\beta})^3}
\end{aligned} \tag{2.27}$$

The remaining two terms are

$$\begin{aligned} &\frac{qR}{y\cdot u}\left(\frac{\hat{n}a^0-\vec{a}}{y\cdot u}-\frac{\gamma y\cdot a}{(y\cdot u)^2}(\hat{n}-\vec{\beta})\right) \\ &=\frac{qR}{(y\cdot u)^3}\left((\hat{n}a^0-\vec{a})(y\cdot u)-\gamma y\cdot a(\hat{n}-\vec{\beta})\right) \end{aligned} \tag{2.28}$$

Working out all of the derivatives, the four-acceleration vector has the form

$$a^\alpha=\left(\gamma^4\left(\dot{\beta}\cdot\beta\right),\gamma^2\dot{\beta}^2+\gamma^4\left(\dot{\beta}\cdot\beta\right)\vec{\beta}\right) \tag{2.29}$$

$$y\cdot a=\gamma^2 R\left[\gamma^2\left(\dot{\beta}\cdot\beta\right)-\hat{n}\cdot\dot{\vec{\beta}}-\gamma^2\left(\dot{\beta}\cdot\beta\right)\left(\hat{n}\cdot\vec{\beta}\right)\right] \tag{2.30}$$

We can now put these into the expression we found above:

$$\begin{aligned} &\frac{qR}{y\cdot u}\left(\frac{\hat{n}a^0-\vec{a}}{y\cdot u}-\frac{\gamma y\cdot a}{(y\cdot u)^2}(\hat{n}-\vec{\beta})\right) \\ &=\frac{qR}{(\gamma R)^3\left(1-\vec{\beta}\cdot\hat{n}\right)^3}\left[\left(\gamma^4\hat{n}\left(\dot{\beta}\cdot\beta\right)-\gamma^2\dot{\vec{\beta}}-\gamma^4\left(\dot{\beta}\cdot\beta\right)\vec{\beta}\right)\gamma R(1-\vec{\beta}\cdot\hat{n})\right. \\ &\left.-\gamma^3 R\left[\gamma^2\left(\dot{\beta}\cdot\beta\right)-\hat{n}\cdot\dot{\vec{\beta}}-\gamma^2\left(\dot{\beta}\cdot\beta\right)\left(\hat{n}\cdot\vec{\beta}\right)\right](\hat{n}-\vec{\beta})\right] \\ &=\frac{q}{R(1-\vec{\beta}\cdot\hat{n})}\left[\gamma^2\left(\dot{\beta}\cdot\beta\right)\left(1-\vec{\beta}\cdot\hat{n}\right)(\hat{n}-\vec{\beta})-\dot{\vec{\beta}}(1-\hat{n}\cdot\vec{\beta})\right. \\ &\left.-\gamma^2\left(\dot{\beta}\cdot\beta\right)\left(1-\vec{\beta}\cdot\hat{n}\right)(\hat{n}-\vec{\beta})+(\hat{n}\cdot\dot{\vec{\beta}})(1-\hat{n}\cdot\vec{\beta})\right] \\ &=\frac{q}{R(1-\vec{\beta}\cdot\hat{n})}\left[(\hat{n}\cdot\dot{\vec{\beta}})(\hat{n}-\vec{\beta})-\dot{\vec{\beta}}\left(1-\hat{n}\cdot\vec{\beta}\right)\right] \end{aligned} \tag{2.31}$$

$$=\frac{q}{R}\frac{\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^3} \tag{2.32}$$

where in the last step we have used the BAC-CAB rule to re-write our expression in terms of a triple cross-product. With this, the electric field indeed becomes the correct form:

$$\vec{E}=\frac{q}{(\gamma R)^2}\frac{\hat{n}-\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})^3}+\frac{q}{R}\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^3} \tag{2.33}$$

Exercise from The Lost Jackson Codex, Vol. XIV

For any field point x not on the world-line $\zeta(t)$ of a (non-tachyonic) particle, show that there is a unique t_0 and $y(t_0)=x-\zeta(t_0)$ such that $y(t_0)\cdot y(t_0)=0$ and $y^0(t_0)>0$.

Due to Veit Elser

Let's assume there are two points with this property such that $y(t_1) \cdot y(t_1) = 0$ and $y(t_2) \cdot y(t_2) = 0$. We can parametrize them as

$$y(t_1) = (R_1, R_1 \hat{n}_1) \qquad y(t_2) = (R_2, R_2 \hat{n}_2) \tag{3.34}$$

where $\hat{n}_1, \hat{n_2}$ are unit vectors. I used the fact $y^0(t_1), y^0(t_2) > 0$ by assuming $R_1, R_2 > 0$. Now let's consider the norm of $y(t_1) - y(t_2)$,

$$(y(t_1) - y(t_2)) \cdot (y(t_1) - y(t_2)) = -2y(t_1) \cdot y(t_2) = -2R_1 R_2 (1 - \hat{n_1} \cdot \hat{n}_2) \leq 0 \tag{3.35}$$

Note that $y(t_1) - y(t_2) = \zeta(t_1) - \zeta(t_2)$.
However, this implies that there are two points on the world-line of the particle $\zeta(t_1), \zeta(t_2)$ that the line between these two points are space-like or null which is a contradiction since any two points on a world-line should be joined by a timelike vector.

Further comment:
Note that we assumed if there are two null intersections with world-line with the condition $y^0(t_1), y^0(t_2) > 0$, we can derive a contradiction. In order to show at least one intersection point exists, one need to use the fact that the **supremum** velocity of a particle will be **smaller than speed of light and not equal to speed of light.** It is a well-known fact that for a particle moving with constant acceleration in special relativity, there is a point x that lightcones emanating from that point never intersect with the world-line of the accelerating particle since the particle asymptotically moves with the speed of light.