## Physics 6561 Fall 2017 Problem Set 7 Solutions

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## Levi-Civita tensor

The standard Lorentz transformation rule — one  $\Lambda$  per index — also applies to the Levi-Civita tensor:

$$(\epsilon')^{\alpha'\beta'\gamma'\delta'} = \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta}.$$

Show that up to overall sign, the Levi-Civita tensor is Lorentz invariant, i.e.  $\epsilon' = \pm \epsilon$ . Hints:

- Express the determinant of a general  $4 \times 4$  matrix  $A^{\alpha}{}_{\beta}$  in terms of Greek indices and  $\epsilon$ .
- $\det AB = \det A \det B$
- $\Lambda^{\alpha}{}_{\beta} \Lambda_{\alpha}{}^{\gamma} = \delta_{\beta}{}^{\gamma}.$

We'd like to show  $(\epsilon')^{\alpha'\beta'\gamma'\delta'} = \pm \epsilon^{\alpha'\beta'\gamma'\delta'}$ .

Before proceeding further, note that symbols like  $\alpha, \beta, \alpha', \gamma', \cdots$  are respresenting a number from the set  $\{0, 1, 2, 3\}$  in 4-dimension and it does not matter if you use them for describing components of  $\epsilon$  tensor or  $\epsilon'$  tensor. However, for the sake of aesthetic reasons, we usually use symbols with prime notation for the components of transformed operators which are also typically denoted by prime notation.

In order to show the result, first note that the right hand side of expression should be proportional to  $\epsilon$  tensor. In other words,

$$\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}\Lambda^{\gamma'}{}_{\gamma}\Lambda^{\delta'}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} = C\epsilon^{\alpha'\beta'\gamma'\delta'}$$
(1.1)

where C is some number. For proving this, we need to show that left hand side of equation 1.1 is totally anti-symmetric tensor. In order to do that, let's exchange  $\alpha'$  and  $\beta'$ :

$$\Lambda^{\beta'}{}_{\alpha} \Lambda^{\alpha'}{}_{\beta} \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} \xrightarrow{\text{relabling } \alpha \text{ and } \beta}{} \Lambda^{\beta'}{}_{\beta} \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} \epsilon^{\beta\alpha\gamma\delta}$$

$$= -\Lambda^{\beta'}{}_{\beta} \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = -\Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} \Lambda^{\gamma'}{}_{\gamma} \Lambda^{\delta'}{}_{\delta} \epsilon^{\alpha\beta\gamma\delta}$$
(1.2)

So that means  $\epsilon'$  is antisymmetric under the exchange of  $\alpha'$  and  $\beta'$ . We chould have chosen any other two indices in order to show  $\epsilon'$  is antisymmetric under their exchange. The only remaining part is to determine the value for C. For finding C, we can set  $\alpha' = 0, \beta' = 1, \gamma' = 2, \delta' = 3$ ,

$$(\epsilon')^{0123} = C\epsilon^{0123} = C = \Lambda^0{}_{\alpha}\Lambda^1{}_{\beta}\Lambda^2{}_{\gamma}\Lambda^3{}_{\delta}\epsilon^{\alpha\beta\gamma\delta}$$
(1.3)

But the right hand side can be quickly be identified as the determinant of  $\Lambda$ . Note that for  $n \times n$  matrix  $A = [A]_{ij}$ ,

$$\det A = \sum_{i_1, i_2, \cdots i_n = 1}^n \epsilon_{i_1 i_2 \cdots i_n} A_{1i_1} A_{2i_2} \cdots A_{i_n n}$$
(1.4)

So by writing equation 1.3 in usual sum notation we find:

$$\Lambda^{0}{}_{\alpha}\Lambda^{1}{}_{\beta}\Lambda^{2}{}_{\gamma}\Lambda^{3}{}_{\delta}\epsilon^{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta}\Lambda^{0\alpha}\Lambda^{1\beta}\Lambda^{2\gamma}\Lambda^{3\delta} = \det(\Lambda\eta) = \det(\Lambda)\det(\eta) = -\det\Lambda$$
(1.5)

Since  $[\Lambda \eta]_{\mu\alpha} = (\Lambda \eta)^{\mu\alpha} = \Lambda^{\mu}{}_{\nu}\eta^{\nu\alpha}$ 

In addition, Lorentz transformations are those real transformations which are satisfying the following relation,

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta^{\alpha\beta} = \eta^{\mu\nu} \qquad \Lambda\eta\Lambda^{T} = \eta$$
$$(\det\Lambda)^{2} = 1 \qquad \Rightarrow \quad \det\Lambda = \pm 1 \tag{1.6}$$

where T denotes transport of the matrix. I as lo used the fact that  ${\rm det}\eta=-1$  and  ${\rm det}\Lambda^T={\rm det}\Lambda.$ 

Combing these together we find:

$$C = -\det \Lambda = \pm 1$$
  

$$\Rightarrow \qquad (\epsilon')^{\alpha'\beta'\gamma'\delta'} = \pm \epsilon^{\alpha'\beta'\gamma'\delta'} \qquad (1.7)$$

The field tensor and its dual

1. Express the Lorentz scalars

$$F^{\alpha\beta}F_{\alpha\beta}, \quad F^{\alpha\beta}\tilde{F}_{\alpha\beta}, \quad \tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta},$$

in terms of  ${\bf E}$  and  ${\bf B}.$ 

Let's review the relationship between  $F_{\alpha\beta}$  and  $\mathbf{E}, \mathbf{B}$ :

$$F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$$

$$F^{0i} = E^{i} \qquad F^{jk} = -\epsilon^{ijk}B^{i}$$

$$F_{0i} = \partial_{0}A_{i} - \partial_{i}A_{0} = -E_{i}$$

$$F_{jk} = -\epsilon_{jki}B^{i} \qquad (1.8)$$

 $F^{\alpha\beta}F_{\alpha\beta}$ :

$$F^{\alpha\beta}F_{\alpha\beta} = F^{0i}F_{0i} + F^{i0}F_{i0} + F^{ij}F_{ij} = -2F^{0i}F^{0i} + F^{ij}F^{ij}$$
  
=  $-2E^{i}E^{i} + (-\epsilon^{ijk}B^{k})(-\epsilon^{ijm}B^{m}) = -2E^{i}E^{i} + (\delta^{jj}\delta^{mk} - \delta^{jm}\delta^{jk})B^{m}B^{k}$   
=  $-2E^{i}E^{i} + 2B^{i}B^{i} = -2\mathbf{E}^{2} + 2\mathbf{B}^{2}$  (1.9)

 $\tilde{F}^{\alpha\beta}F_{\alpha\beta}$ :

$$\tilde{F}^{\alpha\beta}F_{\alpha\beta} = \tilde{F}_{\alpha\beta}F^{\alpha\beta} = 2\tilde{F}_{0i}F^{0i} + \tilde{F}_{ij}F^{ij} 
2F^{0i}(\frac{1}{2}\epsilon_{0ijk})F^{jk} + \frac{1}{2}\epsilon_{ij\mu\nu}F^{\mu\nu}F^{ij} = F^{0i}\epsilon_{0ijk}F^{jk} + \epsilon_{ijk0}F^{k0}F^{ij} = 
F^{0i}\epsilon_{0ijk}F^{jk} + \epsilon_{0kij}F^{k0}F^{ij} = 2F^{0i}\epsilon_{0ijk}F^{jk} 
= 2(E^{i})(-2B^{i}) = -4E^{i}B^{i} = -4\mathbf{E}\cdot\mathbf{B}$$
(1.10)

Where I used  $\epsilon_{0ijk}F^{jk} = \epsilon_{ijk}F^{jk} = -\epsilon_{ijk}\epsilon^{jkm}B^m = -2B^m$ .

 $\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$ :

We don't need to compute this term explicitly again since for  $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ , the result for  $F^{\mu\nu}F_{\mu\nu}$  can be used directly by considering transformations  $\mathbf{E} \to \mathbf{B}$  and  $\mathbf{B} \to -\mathbf{E}$ . So the result is given by

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B})^2$$
(1.11)

2. Curious fact:

 $F^{\alpha\beta}\tilde{F}_{\alpha\beta} = \partial_{\gamma}V^{\gamma},$ 

for some 4-vector quantity  $V^{\gamma}$ . Find  $V^{\gamma}$ .

Let's rewrite  $F^{\alpha\beta}\tilde{F}_{\alpha\beta}$  in terms of 4-potential

$$\begin{split} F^{\alpha\beta}\tilde{F}_{\alpha\beta} &= \frac{\epsilon_{\alpha\beta\gamma\delta}}{2} (\partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha})(\partial^{\gamma}A^{\delta} - \partial^{\delta}A^{\gamma}) \\ &= \frac{\epsilon_{\alpha\beta\gamma\delta}}{2} \partial^{\alpha}A^{\beta}\partial^{\gamma}A^{\delta} - \frac{\epsilon_{\alpha\beta\gamma\delta}}{2} \partial^{\alpha}A^{\beta}\partial^{\delta}A^{\gamma} - \frac{\epsilon_{\alpha\beta\gamma\delta}}{2} \partial^{\beta}A^{\alpha}\partial^{\gamma}A^{\delta} + \frac{\epsilon_{\alpha\beta\gamma\delta}}{2} \partial^{\beta}A^{\alpha}\partial^{\delta}A^{\gamma} \\ &= \frac{\epsilon_{\alpha\beta\gamma\delta}}{2} \partial^{\alpha}A^{\beta}\partial^{\gamma}A^{\delta} - \frac{\epsilon_{\alpha\beta\delta\gamma}}{2} \partial^{\alpha}A^{\beta}\partial^{\gamma}A^{\delta} - \frac{\epsilon_{\beta\alpha\gamma\delta}}{2} \partial^{\alpha}A^{\beta}\partial^{\gamma}A^{\delta} + \frac{\epsilon_{\beta\alpha\delta\gamma}}{2} \partial^{\alpha}A^{\beta}\partial^{\gamma}A^{\delta} \\ &= 2\epsilon_{\alpha\beta\gamma\delta}\partial^{\alpha}A^{\beta}\partial^{\gamma}A^{\delta} = \partial^{\gamma} \left(2\epsilon_{\alpha\beta\gamma\delta}(\partial^{\alpha}A^{\beta})A^{\delta}\right) - 2\epsilon_{\alpha\beta\gamma\delta}(\partial^{\gamma}\partial^{\alpha}A^{\beta})A^{\delta} \\ \partial^{\gamma} \left(2\epsilon_{\alpha\beta\gamma\delta}(\partial^{\alpha}A^{\beta})A^{\delta}\right) &= \partial^{\gamma}V_{\gamma} = \partial_{\gamma}V^{\gamma} \\ V_{\gamma} &= 2\epsilon_{\gamma\delta\alpha\beta}A^{\delta}\partial^{\alpha}A^{\beta} \end{split}$$
(1.12)

Where in third line I relabled indices, for instance in second term I exchanged  $\gamma \Leftrightarrow \delta$ . This can be always done with dummy indices, because we are summing over them and it does not matter what we call them. In the fourth line, I used antisymmetry property of  $\epsilon$  tensor. Second term in the fourth term is zero since  $\epsilon$  is antisymmetric and  $\partial^{\gamma}\partial^{\alpha}A^{\beta}$  is symmetric under  $\gamma \Leftrightarrow \alpha$ . Note that the choice of  $V^{\gamma}$  is not unique, we always have the freedom to add a divergenceless vector to  $V^{\gamma}$ .

if 
$$W^{\gamma}$$
 satisfies  $\partial_{\gamma}W^{\gamma} = 0 \Rightarrow \partial_{\gamma}(V^{\gamma} + W^{\gamma}) = \partial_{\gamma}V^{\gamma}$  (1.13)

3. Is  $V^{\gamma}$  gauge invariant?

We need to determine how 
$$V^{\gamma}$$
 changes under sending  $A_{\mu} \to A_{\mu} + \partial_{\mu}\lambda$ ,  

$$2\epsilon_{\gamma\delta\alpha\beta} \left(A^{\delta} + \partial^{\delta}\lambda\right) \partial^{\alpha} \left(A^{\beta} + \partial^{\beta}\lambda\right) = 2\epsilon_{\gamma\delta\alpha\beta}A^{\delta}\partial^{\alpha}A^{\beta} + 2\epsilon_{\gamma\delta\alpha\beta}\partial^{\delta}\lambda\partial^{\alpha}A^{\beta} + 2\epsilon_{\gamma\delta\alpha\beta}A^{\delta}\partial^{\alpha}\partial^{\beta}\lambda = 2\epsilon_{\gamma\delta\alpha\beta}\partial^{\delta}\lambda\partial^{\alpha}A^{\beta}$$
(1.14)

Where again we used antisymmetry property of  $\epsilon$  and the fact that partial derivatives commute with each other, i.e. symmetric under  $\alpha \Leftrightarrow \beta$ . So this choice of  $V^{\gamma}$  is not gauge invariant.

4. Suppose we modified the action of the electromagnetic field as follows:

$$S[A] = \int d^4x \left(\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \lambda F^{\alpha\beta}\tilde{F}_{\alpha\beta}\right),$$

with some non-zero parameter  $\lambda$ . How would Maxwell's equations be changed? You can answer this question without much work if you take advantage of item 2 above. The Maxwell's equations are invariant by adding this term to Lagrangian, since this term is a total derivative and total derivatives won't affect Euler-Lagrange equation since we can integrate them and they become a boundary term,

$$S[A] = \int d^4x \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) + \lambda \int d^4x \partial_\gamma V^\gamma$$
  
=  $\int d^4x \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right)$   
+  $\lambda \int_{T \to \infty} d^3\vec{x} \left(V^t(T,\vec{x}) - V^t(-T,\vec{x})\right) + \lambda \int_{T \to \infty} d^2S dt \vec{n} \cdot \vec{V}(t,r\vec{n})$  (1.15)

where boundary terms are computed on a boundary of large cylinder on infinity, vanishing by assuming fields die off at large distances and times.

Note that just like classical mechanics, in variational problems, we always fixing value of fields in some given reference times, so we don't really need to assume about behavior of fields at +T, -T.

Moreover, computing the part of equation of motion for this piece of Lagrangian is explicitly given by

$$\delta L = 2\epsilon_{\nu\beta\gamma\delta}\partial^{\nu}A^{\beta}\partial^{\gamma}A^{\delta}$$
$$\frac{\partial\delta L}{\partial A^{\alpha}} - \partial^{\mu}\left(\frac{\partial\delta L}{\partial(\partial^{\mu}A^{\alpha})}\right) = -\partial^{\mu}\left(2\epsilon_{\nu\beta\gamma\delta}(\delta^{\nu}_{\mu}\delta^{\beta}_{\alpha}\partial^{\gamma}A^{\delta} + \delta^{\gamma}_{\mu}\delta^{\delta}_{\alpha}\partial^{\nu}A^{\beta})\right)$$
$$= -4\epsilon_{\mu\alpha\gamma\delta}\partial^{\mu}\partial^{\gamma}A^{\delta} = 0 \tag{1.16}$$

5. Construct all possible cubic Lorentz invariants from F and  $\tilde{F}$ .

## solution due to Alen Senanian:

Any cubic contractions of F and  $\tilde{F}$  must have one of the following forms

$$F^{\mu}{}_{\mu}F^{\alpha\beta}F_{\alpha\beta} \qquad F^{\mu}{}_{\mu}F^{\mu}{}_{\mu}F^{\mu}{}_{\mu} \qquad F^{\mu\alpha}F_{\alpha\beta}F^{\beta}{}_{\mu}, \qquad (1.17)$$

with permuted replacements of F with  $\tilde{F}$  across each of the three expressions. The first two are trivially zero since F and  $\tilde{F}$  are anti-symmetric and thus traceless. To evaluate the third form of the cubic Lorentz invariant, we note that for any anti-symmetric 2nd-rank tensors A and B, we can define a tensor  $S^{\mu}{}_{\nu} = A^{\mu\alpha}B_{\alpha\nu}$  such that

$$S^{\nu}{}_{\mu} = (S^{\mu}{}_{\nu})^{T} = (B_{\alpha\nu})^{T} (A^{\mu\alpha})^{T} = B_{\nu\alpha}A^{\alpha\mu}$$
$$B_{\alpha\nu}A^{\mu\alpha} = S^{\mu}{}_{\nu}$$
(1.18)

That is, the construction of a second rank tensor by the contraction of one degree of freedom from two second rank anti-symmetric tensors is a symmetric one. Further, the full contraction of a symmetric tensor with an anti-symmetric one vanishes

$$S^{\mu}{}_{\nu}A^{\nu}{}_{\mu} = \frac{1}{2} (S^{\mu}{}_{\nu}A^{\nu}{}_{\mu} + S^{\mu}{}_{\nu}A^{\nu}{}_{\mu})$$
  
=  $\frac{1}{2} (S^{\mu}{}_{\nu}A^{\nu}{}_{\mu} + S^{\nu}{}_{\mu}A^{\mu}{}_{\nu})$  (1.19)

where we took advantage of the freedom to relabel any dummy indices for the second term in the last line. Now, since S is symmetric and A is antisymmetric, we have

$$S^{\mu}{}_{\nu}A^{\nu}{}_{\mu} = \frac{1}{2}(S^{\mu}{}_{\nu}A^{\nu}{}_{\mu} - S^{\mu}{}_{\nu}A^{\nu}{}_{\mu}) = 0$$
(1.20)

Therefore, an cubic Lorentz invariant with three anti-symmetric tensors always vanish.

## Further comment:

Another way to see this fact is by considering E and B. Note that a necessary condition for Lorentz invariance is rotational invariance, however, there are no cubic rotational invariant structures made out of electric field and magnetic field. Any cubic rotational invariant construction has the form  $\epsilon_{ijk}E_iE_kB_j$ ,  $\epsilon_{ijk}B_iB_kB_j$ ,... in order to have no free indices which guarantees rotational invariance. However, any of these combinations have at least two componet of E or B, so they vanish due to antisymmetry property of  $\epsilon$  and exchanging symmetic components like  $E_iE_k = E_kE_i$  and so forth.