Physics 6561 Fall 2017
Problem Set 10 Solutions

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Angular momentum content of the magnetic field of a trap

A static magnetic field $\mathbf{B}$, such as used in an atom-trap, can always be expressed as $\nabla\Phi$, where the magnetic potential function Φ satisfies the Laplace equation. A systematic expansion of Φ uses the fact that the functions

$$x \pm iy = r\sin\theta\, e^{\pm i\phi}$$
$$z = r\cos\theta$$

are a basis for the angular momentum functions for $l = 1$. By the addition rule of angular momentum, all angular momentum functions (spherical harmonics) with angular momentum l and below can therefore be expressed as polynomials in x, y, and z of degree at most l.

Show that a Φ of polynomial degree 2 ($l \leq 2$) can never produce a magnetic field having the necessary properties of a magnetic trap:

- $|\mathbf{B}|$ has a local minimum[1].
- At the minimum, $|\mathbf{B}| > 0$.

Φ is a polynomial degree 2 in terms of coordinates x, y, z. Compactly, we can write Φ as

$$\Phi = \mathbf{x}^T A\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{x} + \Phi_0 \tag{1.1}$$

Where A is a matrix characterizing second order terms, $\mathbf{c}$ is determining the first order polynomial in Φ, and Φ_0 is a constant. $\mathbf{x} = \{x, y, z\}$. It is implicit in this expansion that we chose origin at the point in the space that we're determining its magentic filed property, otherwise we should replace $\mathbf{x}$ by $\mathbf{x} - \mathbf{x_0}$.
Using this expression we can also write $\mathbf{B}$ compactly as

$$B = \nabla\Phi = 2A\mathbf{x} + \mathbf{c} \tag{1.2}$$

$$\mathbf{B}(x, y, z) = \mathbf{c} + 2A\mathbf{x} \tag{1.3}$$

[1] Here "local minimum" means the minimum occurs at a point as opposed to an entire line or plane.

Note that if this matrix A is a full rank matrix (the kernel is zero), it should have an inverse. Thus, there is a uniques solution for the point in which magnetic field vanishes,

$$\mathbf{x}^\star = -\frac{1}{2}A^{-1}\mathbf{c} \tag{1.4}$$

where I only used invertibility of A and there is no assumption about $\mathbf{c}$. So that means there is point that $|\mathbf{B}|$ has a local minimum but $|\mathbf{B}|$=0. So this case is not compatible with properties mention in the question.
The only other possibility is that the matrix A is not invertible, and therefore has a null subspace. More explicitly, there is a **non-zero** vector $\mathbf{w}$ such that

$$A\mathbf{w} = 0 \tag{1.5}$$

However, that means if there will be any point, let's say $\mathbf{v}$, in which magnetic field magnitude $|\mathbf{B}| = |\mathbf{B}_0 + A\mathbf{v}|$ reaches the minimum value, there is at least a line of points, given by $\mathbf{v} + \alpha\mathbf{w}$, that for any value of $\alpha \in \mathbb{R}$, the magnetic field has the same minimum value and this is inconsistent with having a minimum magnitude only at the isolated points.

Basis change to align the field

The interaction of a spin-1/2 particle with a magnetic field $\mathbf{B}$ is given by the Hamiltonian term

$$\mu\,\mathbf{B}\cdot\boldsymbol{\sigma},$$

where μ is the particle's magnetic dipole moment and $\boldsymbol{\sigma}$ is the vector of Pauli matrices that make up the spin operator. To better understand the trapping power of a designed, spatially varying magnetic field, we make a basis change (dependent on position) so that "spin-up" always corresponds to the direction of the magnetic field. The required unitary transformation is the following

$$U = \frac{1}{\sqrt{2}}\left(\sqrt{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}}\;\mathbb{1} - i\frac{\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}}{\sqrt{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}}}\right),$$

where $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$. By repeatedly using the Pauli-matrix identity (for general vectors $\mathbf{a}$ and $\mathbf{b}$)

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b}\;\mathbb{1} + i\,\mathbf{a}\times\mathbf{b}\cdot\boldsymbol{\sigma},$$

show that

$$U\;\mathbf{B}\cdot\boldsymbol{\sigma}\;U^\dagger = |\mathbf{B}|\sigma_z.$$

U has two terms, so by expanding $U\mathbf{B}\cdot\sigma U^\dagger$, we find 4 different terms. Before expanding, using BAC-CAB identity, we can work out the following expressions:

$$\begin{aligned}(\mathbf{B}\cdot\boldsymbol{\sigma})(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}) &= \Big(\mathbf{B}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{z}})\Big)\,\mathbb{1} + i[\mathbf{B}\times(\hat{\mathbf{b}}\times\hat{\mathbf{z}})]\cdot\boldsymbol{\sigma} \\ &= i\left[\hat{\mathbf{b}}(\mathbf{B}\cdot\hat{\mathbf{z}}) - \hat{\mathbf{z}}|\mathbf{B}|\right]\cdot\boldsymbol{\sigma} = i(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma})(\mathbf{B}\cdot\hat{\mathbf{z}}) - i(\hat{\mathbf{z}}\cdot\boldsymbol{\sigma})|\mathbf{B}| \\ &= i(\hat{\mathbf{B}}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\hat{\mathbf{z}}) - i(\hat{\mathbf{z}}\cdot\boldsymbol{\sigma})|\mathbf{B}|\end{aligned} \tag{1.6}$$

$$(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma})(\mathbf{B}\cdot\boldsymbol{\sigma}) = -(\mathbf{B}\cdot\boldsymbol{\sigma})(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}) \tag{1.7}$$

Where I used $\mathbf{B}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{z}}) = \hat{\mathbf{z}}\cdot(\mathbf{B}\times\hat{\mathbf{b}}) = 0$.

$$\begin{aligned}&\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\sigma}\Big)\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\sigma}\Big) = \Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\Big)\cdot\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\Big)\,\mathbb{1} \\ &= \sin^2(\theta)\mathbb{1} = \Big(1 - (\hat{\mathbf{z}}\cdot\hat{\mathbf{b}})^2\Big)\,\mathbb{1}\end{aligned} \tag{1.8}$$

Where I used $\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\Big)\times\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\Big) = 0$.
θ is the angle between $\mathbf{B}$ and the z-axis.

$$\begin{aligned}&\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\sigma}\Big)(\mathbf{B}\cdot\boldsymbol{\sigma})\Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\sigma}\Big) = \Big((\hat{\mathbf{b}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\sigma}\Big)\,i\,\Big([\mathbf{B}\times(\hat{\mathbf{b}}\times\hat{\mathbf{z}})]\cdot\boldsymbol{\sigma}\Big) = \\ &= i^2\left[(\hat{\mathbf{b}}\times\hat{\mathbf{z}})\times\Big([\mathbf{B}\times(\hat{\mathbf{b}}\times\hat{\mathbf{z}})]\Big)\right]\cdot\boldsymbol{\sigma} = -\sin^2(\theta)\mathbf{B}\cdot\boldsymbol{\sigma}\end{aligned} \tag{1.9}$$

Now, we need to expand $U\mathbf{B}\cdot\boldsymbol{\sigma}U^\dagger$ and substitute terms by above calculations:

$$\begin{aligned}U\mathbf{B}\cdot\boldsymbol{\sigma}U^\dagger =& \frac{1}{2}\Big(1+\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}\Big)\,\mathbf{B}\cdot\boldsymbol{\sigma} \\ &+ \frac{i}{2}\mathbf{B}\cdot\boldsymbol{\sigma}\,\Big(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}\Big) \\ &- \frac{i}{2}\Big(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}\Big)\,\mathbf{B}\cdot\boldsymbol{\sigma} \\ &+ \frac{1}{2}\frac{1}{1+\hat{\mathbf{b}}\cdot\hat{\mathbf{z}}}\Big(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}\Big)\,\mathbf{B}\cdot\boldsymbol{\sigma}\,\Big(\hat{\mathbf{b}}\times\hat{\mathbf{z}}\cdot\boldsymbol{\sigma}\Big) \\ &= \frac{1}{2}\left(1+\cos(\theta)\right)\mathbf{B}\cdot\boldsymbol{\sigma} + i\,\Big(i[\boldsymbol{\sigma}\cdot\hat{\mathbf{B}}\cos(\theta) - \sigma_z|\mathbf{B}|]\Big) \\ &+ \frac{1}{2(1+\cos\theta)}(-\sin^2(\theta)\mathbf{B}\cdot\boldsymbol{\sigma}) \\ &= (\mathbf{B}\cdot\boldsymbol{\sigma})\frac{(1+\cos\theta)^2 - \sin^2\theta - 2\cos\theta(1+\cos\theta)}{2(1+\cos(\theta))} + |\mathbf{B}|\sigma_z \\ &= |\mathbf{B}|\sigma_z\end{aligned} \tag{1.10}$$

where one can show easily another term is equal to zero and this completes the proof.

Artificial vector potential for the magnetic trap[2]

After making the basis change in the previous problem, the Hamiltonian for the spin-1/2 particle is given by

$$H = -\frac{\hbar^2}{2m}(\nabla + \mathbf{A}) \cdot (\nabla + \mathbf{A}) + \mu\,|\mathbf{B}|\sigma_z$$
$$= -\frac{\hbar^2}{2m}\nabla^2 + \mu\,|\mathbf{B}|\sigma_z + H_{\text{int}},$$

where m is the particle mass and

$$\mathbf{A} = U(\nabla\, U^\dagger)$$

is an artificial vector potential, each vector-component of which is a spin operator (a combination of Pauli matrices). If we could neglect H_{int}, then a spin-up particle would be permanently trapped in a trap where $|\mathbf{B}|$ satisfies the properties given in the first problem. The interaction

$$H_{\text{int}} = -\frac{\hbar^2}{2m}((\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \mathbf{A} + 2\mathbf{A} \cdot \nabla)$$
$$= F + \mathbf{G} \cdot \nabla$$

has two terms that could bring about a transition from spin-up to spin-down. Calculate F and $\mathbf{G}$, **keeping only the lowest non-vanishing terms** in an expansion in powers of x, y and z (i.e. the combined power of r, the distance to the center of the trap). Use the three-parameter expression for $\mathbf{B}$ derived in lecture:

$$\mathbf{B} = \left(\beta x - \frac{\gamma}{2}xz\right)\hat{\mathbf{x}} + \left(-\beta y - \frac{\gamma}{2}yz\right)\hat{\mathbf{y}} + \left(B - \frac{\gamma}{4}(x^2 + y^2) + \frac{\gamma}{2}z^2\right)\hat{\mathbf{z}},$$

B, β and γ are positive and satisfy

$$\frac{\beta^2}{B} > \frac{\gamma}{2}.$$

You should pay special attention to the off-diagonal terms, as they are responsible for the transition.

See attached Mathematica notebook on the website!

[2] You may use *Mathematica* for this problem.