Angular momentum content of the magnetic field of a trap

A static magnetic field $B$, such as used in an atom-trap, can always be expressed as $\nabla \Phi$, where the magnetic potential function $\Phi$ satisfies the Laplace equation. A systematic expansion of $\Phi$ uses the fact that the functions

$$x \pm iy = r \sin \theta e^{\pm i\phi}$$
$$z = r \cos \theta$$

are a basis for the angular momentum functions for $l = 1$. By the addition rule of angular momentum, all angular momentum functions (spherical harmonics) with angular momentum $l$ and below can therefore be expressed as polynomials in $x$, $y$, and $z$ of degree at most $l$.

Show that a $\Phi$ of polynomial degree 2 ($l \leq 2$) can never produce a magnetic field having the necessary properties of a magnetic trap:

- $|B|$ has a local minimum$^1$.
- At the minimum, $|B| > 0$.

\[\Phi = x^T A x + c^T x + \Phi_0 \tag{1.1}\]

Where $A$ is a matrix characterizing second order terms, $c$ is determining the first order polynomial in $\Phi$, and $\Phi_0$ is a constant. $x = \{x, y, z\}$. It is implicit in this expansion that we chose origin at the point in the space that we’re determining its magnetic filed property, otherwise we should replace $x$ by $x - x_0$.

Using this expression we can also write $B$ compactly as

\[B = \nabla \Phi = 2Ax + c \tag{1.2}\]

\[B(x, y, z) = c + 2Ax \tag{1.3}\]

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$^1$Here “local minimum” means the minimum occurs at a point as opposed to an entire line or plane.
Note that if this matrix $A$ is a full rank matrix (the kernel is zero), it should have an inverse. Thus, there is a unique solution for the point in which magnetic field vanishes,

$$\mathbf{x}^* = -\frac{1}{2} A^{-1} \mathbf{c} \quad (1.4)$$

where I only used invertibility of $A$ and there is no assumption about $\mathbf{c}$. So that means there is a point that $|\mathbf{B}|$ has a local minimum but $|\mathbf{B}|=0$. So this case is not compatible with properties mentioned in the question.

The only other possibility is that the matrix $A$ is not invertible, and therefore has a null subspace. More explicitly, there is a **non-zero** vector $\mathbf{w}$ such that

$$A \mathbf{w} = 0 \quad (1.5)$$

However, that means if there will be any point, let’s say $\mathbf{v}$, in which magnetic field magnitude $|\mathbf{B}| = |\mathbf{B}_0 + A\mathbf{v}|$ reaches the minimum value, there is at least a line of points, given by $\mathbf{v} + \alpha \mathbf{w}$, that for any value of $\alpha \in \mathbb{R}$, the magnetic field has the same minimum value and this is inconsistent with having a minimum magnitude only at the isolated points.

**Basis change to align the field**

The interaction of a spin-1/2 particle with a magnetic field $\mathbf{B}$ is given by the Hamiltonian term

$$\mu \mathbf{B} \cdot \sigma,$$

where $\mu$ is the particle’s magnetic dipole moment and $\sigma$ is the vector of Pauli matrices that make up the spin operator. To better understand the trapping power of a designed, spatially varying magnetic field, we make a basis change (dependent on position) so that “spin-up” always corresponds to the direction of the magnetic field. The required unitary transformation is the following

$$U = \frac{1}{\sqrt{2}} \left( \sqrt{1 + \hat{\mathbf{b}} \cdot \hat{\mathbf{z}}} \mathbb{1} - i \frac{\hat{\mathbf{b}} \times \hat{\mathbf{z}} \cdot \sigma}{\sqrt{1 + \hat{\mathbf{b}} \cdot \hat{\mathbf{z}}}} \right),$$

where $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$. By repeatedly using the Pauli-matrix identity (for general vectors $\mathbf{a}$ and $\mathbf{b}$)

$$(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = \mathbf{a} \cdot \mathbf{b} \mathbb{1} + i \mathbf{a} \times \mathbf{b} \cdot \sigma,$$

show that

$$U \mathbf{B} \cdot \sigma U^\dagger = |\mathbf{B}| \sigma_z.$$
$U$ has two terms, so by expanding $UB \cdot \sigma U^\dagger$, we find 4 different terms. Before expanding, using BAC-CAB identity, we can work out the following expressions:

$$
(B \cdot \sigma)(\hat{b} \times \hat{z} \cdot \sigma) = \left( B \cdot (\hat{b} \times \hat{z}) \right) \mathbb{I} + i[B \times (\hat{b} \times \hat{z})] \cdot \sigma
$$

$$
= i \left[ B \cdot (\hat{b} \times \hat{z}) \right] \cdot \sigma = i(\hat{b} \cdot \sigma)(B \cdot \hat{z}) - i(\hat{z} \cdot \sigma)|B|
$$

$$
= i(\hat{b} \cdot \sigma)(B \cdot \hat{z}) - i(\hat{z} \cdot \sigma)|B|
$$

(1.6)

$$
(\hat{b} \times \hat{z} \cdot \sigma)(B \cdot \sigma) = -(B \cdot \sigma)(\hat{b} \times \hat{z} \cdot \sigma)
$$

(1.7)

Where I used $B \cdot (\hat{b} \times \hat{z}) = \hat{z} \cdot (B \times \hat{b}) = 0$.

$$
\left( (\hat{b} \times \hat{z}) \cdot \sigma \right) \left( (\hat{b} \times \hat{z}) \cdot \sigma \right) = \left( (\hat{b} \times \hat{z}) \right) \cdot \mathbb{I} = \sin^2(\theta) \mathbb{I} = \left(1 - (\hat{z} \cdot \hat{b})^2\right) \mathbb{I}
$$

(1.8)

Where I used $(\hat{b} \times \hat{z}) \times (\hat{b} \times \hat{z}) = 0$.

$\theta$ is the angle between $B$ and the z-axis.

$$
\left( (\hat{b} \times \hat{z}) \cdot \sigma \right) (B \cdot \sigma) \left( (\hat{b} \times \hat{z}) \cdot \sigma \right) = \left( (\hat{b} \times \hat{z}) \cdot \sigma \right) i \left( [B \times (\hat{b} \times \hat{z})] \cdot \sigma \right) = i^2 \left[ (\hat{b} \times \hat{z}) \times \left([B \times (\hat{b} \times \hat{z})]\right) \right] \cdot \sigma = -\sin^2(\theta)B \cdot \sigma
$$

(1.9)

Now, we need to expand $UB \cdot \sigma U^\dagger$ and substitute terms by above calculations:

$$
UB \cdot \sigma U^\dagger = \frac{1}{2} \left(1 + \hat{b} \cdot \hat{z}\right)B \cdot \sigma
$$

$$
+ \frac{i}{2} B \cdot \sigma \left( \hat{b} \times \hat{z} \cdot \sigma \right)
$$

$$
- \frac{i}{2} \left( \hat{b} \times \hat{z} \cdot \sigma \right) B \cdot \sigma
$$

$$
+ \frac{1}{2} \frac{1}{1 + \hat{b} \cdot \hat{z}} \left( \hat{b} \times \hat{z} \cdot \sigma \right) B \cdot \sigma \left( \hat{b} \times \hat{z} \cdot \sigma \right)
$$

$$
= \frac{1}{2} \left(1 + \cos(\theta)\right)B \cdot \sigma + i \left( i[\sigma \cdot \hat{B} \cos(\theta) - \sigma_z |B|]\right)
$$

$$
+ \frac{1}{2(1 + \cos(\theta))}(-\sin^2(\theta)B \cdot \sigma)
$$

$$
= (B \cdot \sigma) \frac{(1 + \cos\theta)^2 - \sin^2(\theta) - 2 \cos(\theta)(1 + \cos\theta)}{2(1 + \cos(\theta))} + |B|\sigma_z
$$

$$
= |B|\sigma_z
$$

(1.10)
where one can show easily another term is equal to zero and this completes the proof.

Artificial vector potential for the magnetic trap

After making the basis change in the previous problem, the Hamiltonian for the spin-1/2 particle is given by

\[
H = -\frac{\hbar^2}{2m} \nabla \cdot (\nabla + A) + \mu |B| \sigma_z
\]

\[
= -\frac{\hbar^2}{2m} \nabla^2 + \mu |B| \sigma_z + H_{\text{int}},
\]

where \( m \) is the particle mass and

\[
A = U (\nabla U^\dagger)
\]

is an artificial vector potential, each vector-component of which is a spin operator (a combination of Pauli matrices). If we could neglect \( H_{\text{int}} \), then a spin-up particle would be permanently trapped in a trap where \( |B| \) satisfies the properties given in the first problem. The interaction

\[
H_{\text{int}} = -\frac{\hbar^2}{2m} ((\nabla \cdot A) + A \cdot A + 2A \cdot \nabla)
\]

\[
= F + G \cdot \nabla
\]

has two terms that could bring about a transition from spin-up to spin-down. Calculate \( F \) and \( G \), keeping only the lowest non-vanishing terms in an expansion in powers of \( x, y \) and \( z \) (i.e. the combined power of \( r \), the distance to the center of the trap). Use the three-parameter expression for \( B \) derived in lecture:

\[
B = \left( \beta x - \frac{\gamma}{2} x z \right) \hat{x} + \left( -\beta y - \frac{\gamma}{2} y z \right) \hat{y} + \left( B - \frac{\gamma}{4} (x^2 + y^2) + \frac{\gamma}{2} z^2 \right) \hat{z},
\]

\( B, \beta \) and \( \gamma \) are positive and satisfy

\[
\frac{\beta^2}{B} > \frac{\gamma}{2}.
\]

You should pay special attention to the off-diagonal terms, as they are responsible for the transition.

See attached Mathematica notebook on the website!

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\(^2\)You may use Mathematica for this problem.