Physics 6561 Fall 2017
Problem Set 6 Solutions

Amir Tajdini
at734@cornell.edu

Ground state wavefunction of the electromagnetic field

Recall from lecture the Hamiltonian operator for the electromagnetic field,

$$\hat{H} = \int d^3x \left(-\frac{(\hbar c)^2}{2} \partial_{A_i(\mathbf{x})} \partial_{A_i(\mathbf{x})} + \frac{1}{2} |\nabla \times \mathbf{A}(\mathbf{x})|^2 \right),$$

and ground state wave-functional:

$$\Psi_0[\mathbf{A}(\mathbf{x})] = \exp\left(-\kappa \int d^3x \int d^3y \, \frac{\nabla \times \mathbf{A}(\mathbf{x}) \cdot \nabla \times \mathbf{A}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} \right).$$

Your task is to determine the value of the constant κ that is consistent with the Schrödinger equation

$$\hat{H}\Psi_0 = E_0 \Psi_0.$$

You can do this without having to evaluate E_0 (which is infinite in the absence of a cutoff).

As a warm-up, let's do the analogous calculation for the 1D harmonic oscillator with Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \partial_x^2 + \frac{k}{2} x^2$$

and ground state wavefunction $\Psi_0 = \exp(-cx^2)$. The pair of derivatives in $\hat{H}$ acting on Ψ_0 produces two terms:

$$\partial_x^2 \Psi_0 = \left(\partial_x(-cx^2) \right)^2 \Psi_0 + \left(\partial_x \partial_x (-cx^2) \right) \Psi_0.$$

For the correct value of c, the first term exactly cancels the $(k/2)x^2\Psi_0$ term in $\hat{H}\Psi_0$. This is the extent of what you are being asked to do in the case of the electromagnetic field Schrödinger equation. It's enough to note that the second term is independent of x (or $\mathbf{A}(\mathbf{x})$) and therefore its value (even if infinite!) is the energy eigenvalue.

For many of you the main challenge in this problem will be working with the "variational" or "functional" derivative $\partial_{A_i(\mathbf{x})}$. Below are some instances of how this operator acts. Be sure you understand each example! Repeated latin indices are summed, as usual.

In these, $f(\,)$ is an arbitrary scalar function (not functional) taking a vector argument:

$$\partial_{A_i(\mathbf{x})} \partial_{A_i(\mathbf{x})} \, f(\mathbf{A}(\mathbf{y})) = 0, \qquad \mathbf{x} \neq \mathbf{y}$$

$$\partial_{A_i(\mathbf{x})} \partial_{A_i(\mathbf{x})} \, f(\mathbf{A}(\mathbf{x})) = \nabla^2_{\mathbf{w}} f(\mathbf{w}) \big|_{\mathbf{w}=\mathbf{A}(\mathbf{x})}.$$

The following show the action of just a single functional derivative:

$$\partial_{A_i(\mathbf{x})} \left(A_j(\mathbf{x}) A_j(\mathbf{y}) \right) = A_i(\mathbf{y})$$

$$\partial_{A_i(\mathbf{x})} \int d^3y\; B_j(\mathbf{y}) A_j(\mathbf{y}) = B_i(\mathbf{x})$$

$$\begin{aligned} \partial_{A_i(\mathbf{x})} \int d^3y\; f(\mathbf{y}) \nabla \cdot \mathbf{A}(\mathbf{y}) &= \partial_{A_i(\mathbf{x})} \int d^3y\; \left(-\nabla f(\mathbf{y})\right) \cdot \mathbf{A}(\mathbf{y}) \\ &= -\partial_{x_i} f(\mathbf{x}). \end{aligned}$$

As in these examples, you may always neglect boundary terms when integrating by parts.

Here are some concrete hints, roughly in the order you'll need them.

1. $$\nabla_{\mathbf{z}}\, f(|\mathbf{x}-\mathbf{z}|) = -\nabla_{\mathbf{x}}\, f(|\mathbf{x}-\mathbf{z}|)$$
2. $$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$
3. $$\int \frac{d^3z}{|\mathbf{x}-\mathbf{z}|^2 |\mathbf{y}-\mathbf{z}|^2} = \frac{\pi^3}{|\mathbf{x}-\mathbf{y}|}$$
4. $$\nabla \cdot \nabla \times \mathbf{v} = 0$$
5. $$-\nabla_{\mathbf{x}}^2 \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) = 4\pi \delta^3(\mathbf{x}-\mathbf{y})$$

Finally, here are some motivational quotes:

Let's name the argument inside the exponential in the eigenfunction as $F[\mathbf{A}(x)]$:

$$\begin{aligned} F[\mathbf{A}(x)] &= -\kappa \int d^3x \int d^3y\; \frac{\nabla \times \mathbf{A}(\mathbf{x}) \cdot \nabla \times \mathbf{A}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} \\ \Psi_0[\mathbf{A}(x)] &= \exp\left(F[\mathbf{A}(x)]\right) \end{aligned} \tag{1.1}$$

So substituting in Schrödinger equation, similar to harmonic oscillator example we find:

$$\hat{H}\Psi_0[\mathbf{A}(x)] = E_0\Psi_0 = \int d^3x \left(-\frac{(\hbar c)^2}{2} \left(\frac{\partial F}{\partial A_i(x)} \right) \left(\frac{\partial F}{\partial A_i(x)} \right) \Psi_0 \right. \tag{1.2}$$

$$\left. -\frac{(\hbar c)^2}{2} \frac{\partial^2 F}{\partial A_i(x)^2} \Psi_0 + \frac{1}{2} |\nabla \times \mathbf{A}(\mathbf{x})|^2 \Psi_0 \right). \tag{1.3}$$

Since F is second order in $A(x)$, by taking functional derivative of F with respect to $A(x)$ twice, there will be no A dependence left, so in order to satisfy Schrödinger equation, the first term in the Eq 1.2 needs to cancel the third term which assumes E_0 does not

depend on $\mathbf{A(x)}$ as we expect for ground state even if its energy is infinite!
Let's work out details of $\partial_{A_i(x)F}$:

$$\partial_{A_i(x)}F[\mathbf{A}(\mathbf{x})] = -\kappa\partial_{A_i(x)}\int\int d^3y d^3z \frac{\nabla\times\mathbf{A}(\mathbf{y})\cdot\nabla\times\mathbf{A}(\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2}$$
$$= -2\kappa\int\int d^3y d^3z \frac{\left(\partial_{A_i(x)}\nabla\times\mathbf{A}(\mathbf{y})\right)\cdot\nabla\times\mathbf{A}(\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^2} \tag{1.4}$$

Where factors of two coming from symmetry $y \leftrightarrow z$. In other words, when derivative is acting on $\mathbf{A}(\mathbf{z})$ term, we renamed y, z and denominator is symmetric under this exchange and as a result we have symmetry factor equals to 2.

Let's evaluate $\left(\partial_{A_i(x)}\nabla\times\mathbf{A}(\mathbf{y})\right)$:

$$(\nabla\times\mathbf{A}(\mathbf{y}))^l = \varepsilon^l_{mn}\partial^{(y)}_m A_n(y)$$
$$\left(\partial_{A_i(x)}\nabla\times\mathbf{A}(\mathbf{y})\right) = \left(\partial_{A_i(x)}\varepsilon^l_{mn}\partial^{(y)}_m A_n(y)\right) = \epsilon^l_{mn}\partial^{(y)}_m\delta^3(\mathbf{x}-\mathbf{y})\delta_{in}$$
$$= \epsilon^l_{mi}\partial^{(y)}_m\delta^3(x-y) \tag{1.5}$$

substituting back we find:

$$\left(\partial_{A_i(x)}\nabla\times\mathbf{A}(\mathbf{y})\right)\cdot\nabla\times\mathbf{A}(\mathbf{z}) = \left(\partial_{A_i(x)}\nabla\times\mathbf{A}(\mathbf{y})\right)^l\left(\nabla^{(z)}\times\mathbf{A}(\mathbf{z})\right)^l$$
$$= \epsilon^l_{mi}\partial^{(y)}_m\delta^3(\mathbf{x}-\mathbf{y})\epsilon^l_{jk}\partial^{(z)}_j A_k(z) = (\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})\partial^{(y)}_m\delta^3(\mathbf{x}-\mathbf{y})\partial^{(z)}_j A_k(z)$$
$$= \partial^{(y)}_m\delta^3(\mathbf{x}-\mathbf{y})\partial^{(z)}_m A_i(z) - \partial^{(y)}_m\delta^3(\mathbf{x}-\mathbf{y})\partial^{(z)}_i A_m(z) \tag{1.6}$$

Therefore we simplifies numerator in Eq. 1.4

$$\partial_{A_i(x)}F[\mathbf{A}(\mathbf{x})] = -2\kappa \int\int d^3y d^3z \frac{\partial_m^{(y)}\delta^3(\mathbf{x}-\mathbf{y})\partial_m^{(z)}A_i(z) - \partial_m^{(y)}\delta^3(\mathbf{x}-\mathbf{y})\partial_i^{(z)}A_m(z)}{|\mathbf{y}-\mathbf{z}|^2}$$
$$= 2\kappa \int\int d^3y d^3z \delta^3(\mathbf{x}-\mathbf{y})\partial_m^{(z)}A_i(z)\partial_m^{(y)}\frac{1}{|\mathbf{y}-\mathbf{z}|^2}$$
$$= -2\kappa \int\int d^3y d^3z \delta^3(\mathbf{x}-\mathbf{y})\partial_i^{(z)}A_m(z)\partial_m^{(y)}\frac{1}{|\mathbf{y}-\mathbf{z}|^2}$$
$$-2\kappa \int\int d^3y d^3z \delta^3(\mathbf{x}-\mathbf{y})\partial_m^{(z)}A_i(z)\partial_m^{(z)}\frac{1}{|\mathbf{y}-\mathbf{z}|^2} + 2\kappa \int\int d^3y d^3z \delta^3(\mathbf{x}-\mathbf{y})\partial_i^{(z)}A_m(z)\partial_m^{(z)}\frac{1}{|\mathbf{y}-\mathbf{z}|^2}$$
$$-2\kappa \int d^3z \partial_m^{(z)}A_i(z)\partial_m^{(z)}\frac{1}{|\mathbf{x}-\mathbf{z}|^2} + 2\kappa \int d^3z \partial_i^{(z)}A_m(z)\partial_m^{(z)}\frac{1}{|\mathbf{x}-\mathbf{z}|^2}$$
$$2\kappa \int d^3z \partial_m^{(z)}\partial_m^{(z)}A_i(z)\frac{1}{|\mathbf{x}-\mathbf{z}|^2} - 2\kappa \int d^3z \partial_m^{(z)}\partial_i^{(z)}A_m(z)\frac{1}{|\mathbf{x}-\mathbf{z}|^2}$$
$$2\kappa \int d^3z \frac{\nabla^2 A_i(z) - \partial_i(\nabla\cdot\mathbf{A}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2} \tag{1.7}$$

Where I used

$$\int d^3z \partial_m^{(z)}A_i(z)\partial_m^{(z)}\frac{1}{|\mathbf{x}-\mathbf{z}|^2} = -\int d^3z \partial_m^{(z)}\partial_m^{(z)}A_i(z)\frac{1}{|\mathbf{x}-\mathbf{z}|^2}$$

$$\int d^3z \partial_i^{(z)}A_m(z)\partial_m^{(z)}\frac{1}{|\mathbf{x}-\mathbf{z}|^2} = -\int d^3z \partial_m^{(z)}\partial_i^{(z)}A_m(z)\frac{1}{|\mathbf{x}-\mathbf{z}|^2}$$

by assuming $\mathbf{A}(\mathbf{z})$ is falls off quickly at infinity.

Finally,

$$\int d^3x (\partial_{A_i(x)}F)(\partial_{A_i(x)}F)$$
$$= 4\kappa^2 \int d^3x \int d^3z d^3y \frac{{\nabla^{(z)}}^2 A_i(z) - \partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A}(\mathbf{z}))}{|\mathbf{x}-\mathbf{z}|^2}\frac{{\nabla^{(y)}}^2 A_i(y) - \partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A}(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^2}$$
$$= 4\kappa^2 \int d^3z d^3y \left({\nabla^{(z)}}^2 A_i(z) - \partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A}(\mathbf{z}))\right)$$
$$\times \left({\nabla^{(y)}}^2 A_i(y) - \partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A}(\mathbf{y}))\right) \int \frac{d^3x}{|\mathbf{x}-\mathbf{z}|^2|\mathbf{x}-\mathbf{y}|^2}$$

using 3rd hint,

$$= 4\kappa^2\pi^3 \int\int \frac{d^3zd^3y}{|\mathbf{y}-\mathbf{z}|}\left({\nabla^{(z)}}^2 A_i(z) - \partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A(z)})\right)\left({\nabla^{(y)}}^2 A_i(y) - \partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)})\right)$$

$$= 4\kappa^2\pi^3 \int\int \frac{d^3zd^3y}{|\mathbf{y}-\mathbf{z}|}{\nabla^{(z)}}^2 A_i(z){\nabla^{(y)}}^2 A_i(y) - 4\kappa^2\pi^3 \int\int \frac{d^3zd^3y}{|\mathbf{y}-\mathbf{z}|}{\nabla^{(z)}}^2 A_i(z)\partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)})$$

$$- 4\kappa^2\pi^3 \int\int \frac{d^3zd^3y}{|\mathbf{y}-\mathbf{z}|}\partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A(z)}){\nabla^{(y)}}^2 A_i(y)$$

$$+ 4\kappa^2\pi^3 \int\int \frac{d^3zd^3y}{|\mathbf{y}-\mathbf{z}|}\partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A(z)})\partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)})$$

doing integration by parts,

$$\begin{aligned}
&= +4\kappa^2\pi^3 \int\int d^3zd^3y A_i(z){\nabla^{(y)}}^2 A_i(y){\nabla^{(z)}}^2\frac{1}{|\mathbf{y}-\mathbf{z}|}\\
&- 4\kappa^2\pi^3 \int\int d^3zd^3y A_i(z)\partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)}{\nabla^{(z)}}^2\frac{1}{|\mathbf{y}-\mathbf{z}|}\\
&- 4\kappa^2\pi^3 \int\int d^3zd^3y A_i(y)\partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A(z)}{\nabla^{(y)}}^2\frac{1}{|\mathbf{y}-\mathbf{z}|}\\
&+ 4\kappa^2\pi^3 \int\int d^3zd^3y (\nabla^{(z)}\cdot\mathbf{A(z)})(\nabla^{(y)}\cdot\mathbf{A(y)})\partial_i^{(z)}\partial_i^{(y)}\frac{1}{|\mathbf{y}-\mathbf{z}|}
\end{aligned}$$

using 1st and 5th hints,

$$\begin{aligned}
&= +4\kappa^2\pi^3 \int\int d^3zd^3y A_i(z){\nabla^{(y)}}^2 A_i(y)\left(-4\pi\delta^3(\mathbf{y}-\mathbf{z})\right)\\
&- 4\kappa^2\pi^3 \int\int d^3zd^3y A_i(z)\partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)}\left(-4\pi\delta^3(\mathbf{y}-\mathbf{z})\right)\\
&- 4\kappa^2\pi^3 \int\int d^3zd^3y A_i(y)\partial_i^{(z)}(\nabla^{(z)}\cdot\mathbf{A(z)}\left(-4\pi\delta^3(\mathbf{y}-\mathbf{z})\right)\\
&+ 4\kappa^2\pi^3 \int\int d^3zd^3y (\nabla^{(z)}\cdot\mathbf{A(z)})(\nabla^{(y)}\cdot\mathbf{A(y)})\left(4\pi\delta^3(\mathbf{y}-\mathbf{z})\right)
\end{aligned} \tag{1.8}$$

$$\begin{aligned}
&= -16\kappa^2\pi^4 \int d^3y A_i(y){\nabla^{(y)}}^2 A_i(y)\\
&+ 16\kappa^2\pi^4 \int d^3y A_i(y)\partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)})\\
&+ 16\kappa^2\pi^4 \int d^3y A_i(y)\partial_i^{(y)}(\nabla^{(y)}\cdot\mathbf{A(y)})\\
&+ 16\kappa^2\pi^4 \int d^3y (\nabla^{(y)}\cdot\mathbf{A(y)})(\nabla^{(y)}\cdot\mathbf{A(y)}).
\end{aligned}$$

Organizing terms, we find:

$$
\begin{aligned}
&= +16\kappa^2\pi^4 \int d^3y \partial_j A_i(y) \partial_j A_i(y) \\
&- 32\kappa^2\pi^4 \int d^3y \partial_i A_i(y) \partial_j A_j(y) \\
&+ 16\kappa^2\pi^4 \int d^3y \partial_j A_j(y) \partial_i A_i(y)
\end{aligned}
\tag{1.9}
$$

$$
= 16\kappa^2\pi^4 \int d^3y \left(\partial_j A_i(y)\partial_j A_i(y) - \partial_i A_i(y)\partial_j A_j(y)\right) \tag{1.10}
$$

I used the 1st hint explicitly for computing

$$
\partial_i^{(z)} \partial_i^{(y)} \frac{1}{|\mathbf{y} - \mathbf{z}|} = -\partial_i^{(z)} \partial_i^{(z)} \frac{1}{|\mathbf{y} - \mathbf{z}|} = 4\pi\delta^3(\mathbf{y} - \mathbf{z}). \tag{1.11}
$$

Meanwhile, simplifying $|\nabla \times \mathbf{A}|^2$ term in Hamiltonian, we find:

$$
\begin{aligned}
&\int d^3x |\nabla \times \mathbf{A}(\mathbf{x})|^2 = \int d^3x \epsilon_{ijk} \partial_j A_k \epsilon_{imn} \partial_m A_n \\
&\int d^3x (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \partial_j A_k \partial_m A_n \\
&= \int d^3x (\partial_j A_i(x) \partial_j A_i(x) - \partial_i A_i(x) \partial_j A_j(x)).
\end{aligned}
\tag{1.12}
$$

Therefore, equating coefficient of these two terms, we find:

$$
\begin{aligned}
&-\frac{\hbar^2 c^2}{2} 16\kappa^2\pi^4 + \frac{1}{2} = 0 \\
&\Rightarrow \qquad \kappa = \frac{1}{4\pi^2 \hbar c}.
\end{aligned}
\tag{1.13}
$$

Note that negative solution for κ is corresponding to exponentially growing answer.