Physics 6561 Fall 2017
Problem Set 5 Solutions

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Vector field decomposition I

Given the definitions in lecture of $\mathbf{v}_L$ and $\mathbf{v}_T$, for a general vector field $\mathbf{v}$ in three dimensions (no boundaries), show that $\nabla \times \mathbf{v}_L = 0$ and $\nabla \cdot \mathbf{v}_T = 0$.

For curl of $\mathbf{v}$,

$$\nabla \times \mathbf{v}_L = -\nabla \times \left(\nabla \left(\int d^3\mathbf{x}' \frac{\nabla \cdot \mathbf{v}}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) \right) = 0 \tag{1.1}$$

since $\nabla \times (\nabla f) = 0$ for any arbitrary function f.
For divergence of $\mathbf{v}_T$ we have

$$\begin{aligned}
\nabla \cdot \mathbf{v}_T &= \nabla \cdot \mathbf{v} + \nabla \cdot \left(\nabla \left(\int d^3\mathbf{x}' \frac{\nabla_{\mathbf{x}'} \cdot \mathbf{v}}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) \right) \\
&= \nabla \cdot \mathbf{v} + \nabla^2_{\mathbf{x}} \left(\int d^3\mathbf{x}' \frac{\nabla_{\mathbf{x}'} \cdot \mathbf{v}}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) = \\
&= \nabla \cdot \mathbf{v} + \int d^3\mathbf{x}' \left(\nabla_{\mathbf{x}'} \cdot \mathbf{v}\right) \nabla^2_{\mathbf{x}} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \\
&= \nabla \cdot \mathbf{v} + \int d^3\mathbf{x}' \left(\nabla_{\mathbf{x}'} \cdot \mathbf{v}\right) \left(-\delta^3(\mathbf{x} - \mathbf{x}')\right) = 0
\end{aligned} \tag{1.2}$$

So we showed both $\nabla \times \mathbf{v}_L = 0$, $\nabla \cdot \mathbf{v}_T = 0$.

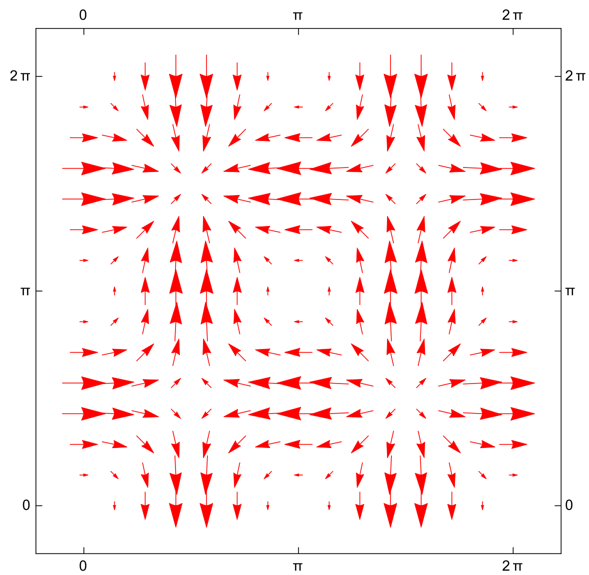
Vector field decomposition II

The figure below shows one period of the following periodic vector field in two dimensions:

$$\mathbf{v} = (v_x, v_y) = (\cos x \, \sin^2 y, -\cos y \, \sin^2 x).$$

Notice that this field has both non-vanishing divergence and non-vanishing circulation: it is neither transverse nor longitudinal.

In this problem you will use the discrete Fourier transform to numerically perform the decomposition $\mathbf{v} = \mathbf{v}_\mathrm{T} + \mathbf{v}_\mathrm{L}$ into transverse and longitudinal fields. If you do not have access to, or experience with, software for computing FFTs or rendering vector fields, team up with someone who does.



The first step is to choose the grid dimension N. The grid points $\mathbf{x}$ have integer coordinates that run from 0 to $N-1$. Next, compute the 2D FFTs of v_x and v_y, sampled on this grid. We can write the relationship between the two representations of the vector fields as follows,

$$\mathbf{v}(\mathbf{x}) = \frac{1}{\left(\sqrt{N}\right)^2} \sum_{\mathbf{k}} e^{i2\pi(\mathbf{k}\cdot\mathbf{x})/N}\, \tilde{\mathbf{v}}(\mathbf{k}), \tag{1.3}$$

where the sum is over another 2D grid of points $\mathbf{k}$ with integer coordinates. This grid also has size $N \times N$ because the phase factor is unchanged when any multiple of N is added to a component of $\mathbf{k}$. However, the choice of $\mathbf{k}$ will affect the decomposition into transverse and longitudinal. As an extreme case consider the $\mathbf{k}$'s $(1,0)$ and $(-1,0)$. Both of these correspond to the smallest possible spatial variation on the grid. However, the second of these is equivalent to $(N-1,0)$ — a very large spatial variation that would only be noticed if one could sample *between* the integer points $\mathbf{x}$. To get the correct continuum limit from our grid samples we should always choose the $\mathbf{k}$ with the smallest magnitude. In practice this means each component of $\mathbf{k}$ runs as $0,1,2,\ldots,-2,-1$ instead of $0,1,2,\ldots,N-2,N-1$.

The longitudinal and transverse projections of $\tilde{\mathbf{v}}$ are defined by:

$$\begin{aligned}
\tilde{\mathbf{v}}_{\mathrm{L}}(\mathbf{k}) &= (\tilde{\mathbf{v}}(\mathbf{k})\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}} \\
\tilde{\mathbf{v}}_{\mathrm{T}}(\mathbf{k}) &= \tilde{\mathbf{v}}(\mathbf{k}) - (\tilde{\mathbf{v}}(\mathbf{k})\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}.
\end{aligned}$$

By replacing $\tilde{\mathbf{v}}$ in (1) with $\tilde{\mathbf{v}}_{\mathrm{L}}$, the resulting vector field will be a superposition of waves whose polarization is always parallel to the direction of propagation ($\hat{\mathbf{k}}$); the perpendicular relationship holds when we use $\tilde{\mathbf{v}}_{\mathrm{T}}$ instead. The last step is therefore to undo the earlier FFTs after making these replacements, thus producing the vector fields $\mathbf{v}_{\mathrm{L}}$ and $\mathbf{v}_{\mathrm{T}}$. Make plots to visually confirm the curl-free and divergence-free properties.

Solution due to Chris Wilson and Daniel Longenecker. See the attached Mathematica file for a sample code.

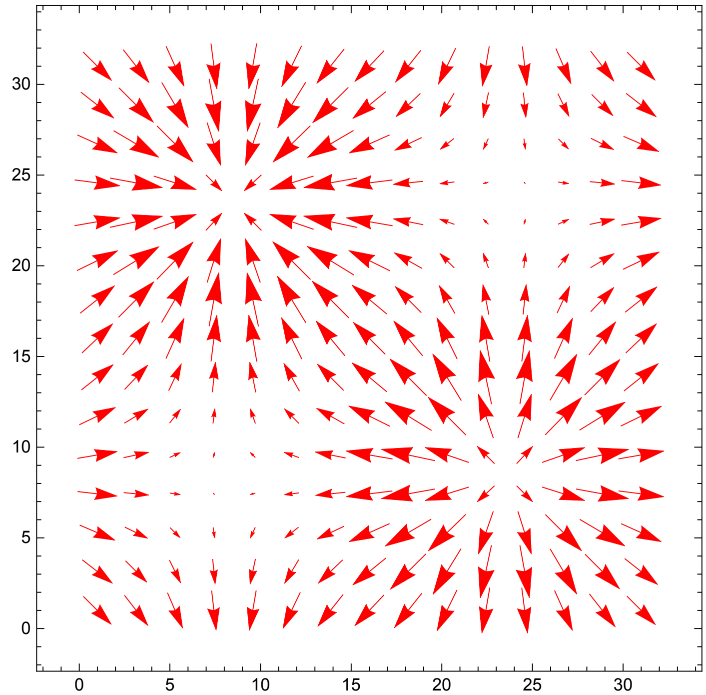


Figure 1: $\mathbf{v}_L$

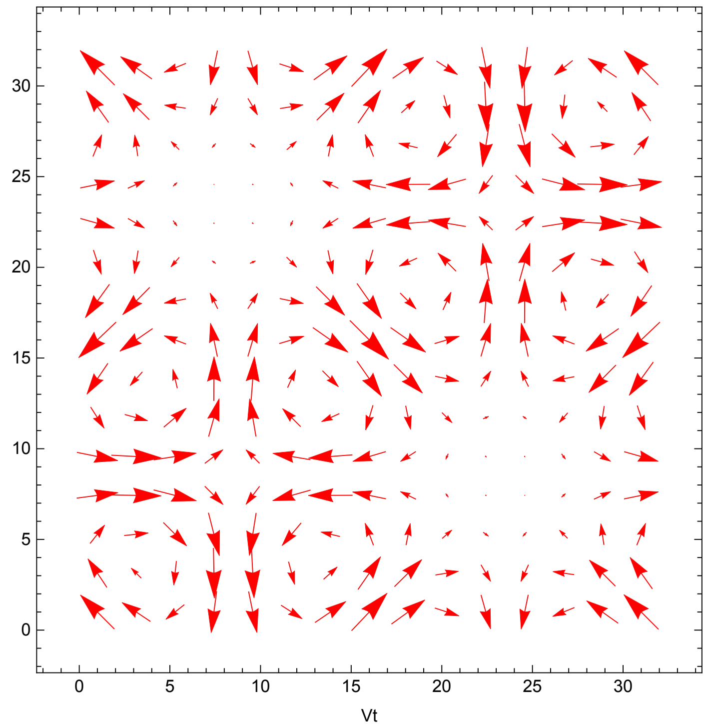


Figure 2: $\mathbf{v}_T$

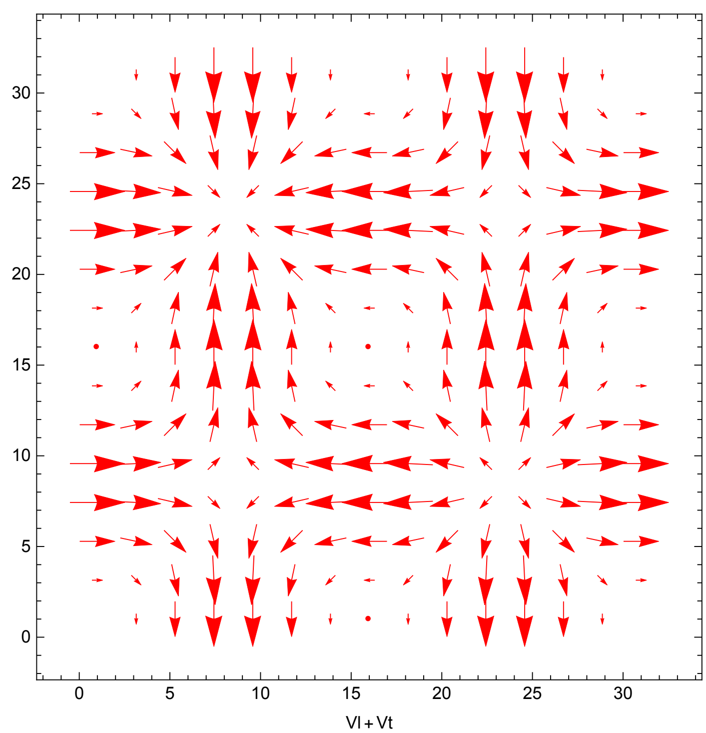


Figure 3: $\mathbf{v}_T + \mathbf{v}_L$

Fields produced by a current sheet

The current density in an infinite sheet has the form

$$\mathbf{J}(\mathbf{x}, t) = c\sigma\delta(z)\cos kx \cos\omega t\ \hat{\mathbf{y}},$$

and the charge density ρ vanishes everywhere at all times (the constant σ has units of surface charge density).

1. Is charge conserved? Is the current density everywhere transverse?

> Using local form of charge conservation
>
> $$\frac{\partial\rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{2.4}$$
>
> and checking divergence of current, we have
>
> $$\nabla \cdot \mathbf{J} = \frac{\partial}{\partial y}\left(c\sigma\delta(z)\cos kx \cos\omega t\right) = 0 \tag{2.5}$$
>
> That means it's consistent with considering $\rho = 0$ everywhere and charge conservation tells us if $\rho = 0$ initially, it will remain zero in later times.

2. Calculate $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ everywhere in space and time; consider separately the cases $\omega < ck$ and $\omega > ck$. Hint: first argue that

$$\mathbf{A}(\mathbf{x}, t) = \mathrm{Re}\left(f(z)e^{-i\omega t}\right)\cos kx\ \hat{\mathbf{y}}$$

is a valid expression for the vector potential in this problem.

> In transverse gauge, Maxwell equation for $\mathbf{A}$ has the form
>
> $$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{4\pi}{c}\mathbf{J}_T \tag{2.6}$$
>
> Since $\rho = 0$, we can assume $V = 0$ throughout the space and time. This in particular implies $\nabla \cdot \mathbf{A} = 0$. $\mathbf{J}$ is in $\hat{y}$ direction, so $\mathbf{A}$ is also pointing in the same direction $\hat{\mathbf{y}}$. Also because $\mathbf{A}$ is divergenceless, there should be no y-dependence in A_y. In addition, knowing the fact that $\mathbf{A}$ can be generally written in terms of combination of retarded and advanced solution in terms of $\mathbf{J}$, one can argue x dependence should be in a form of $\cos kx$. This leads to the guess that the vector potential has the form
>
> $$\mathbf{A} = \mathrm{Re}\left(f(z)e^{-i\omega t}\right)\cos kx\ \hat{\mathbf{y}}. \tag{2.7}$$

Note that since $\mathbf{J}$ is real, we can write it as $\mathrm{Re}\left(c\delta(z)\cos kxe^{-i\omega t}\right)$.
Now by inserting into Maxwell equation 2.6 and removing real parts from both side (remember in general $\mathrm{Re}(z_1z_2) \neq \mathrm{Re}(z_1)\mathrm{Re}(z_2)$), one finds

$$f''(z) + (\frac{\omega^2}{c^2} - k^2)f(z) = -4\pi\sigma\delta(z) \tag{2.8}$$

There are different ways to solve this equation, one can use Fourier transformation and picking correct poles in order to find outgoing radiation. Here we use the simplest approach, resembling solving Schrodinger equation in 1-dimension with δ-function potential.

(a) $k \geq \omega/c$:

General solution for eq. 2.8 is given by:

$$f(z) = Ae^{Qz} + Be^{-Qz} \tag{2.9}$$

Where $Q = \sqrt{k^2 - \frac{\omega^2}{c^2}}$. Because of delta function at origin, we need to separate solution above and below the sheet of current:

$$f(z) = \begin{cases} A\mathrm{exp}(Qz) + B\mathrm{exp}(-Qz) & z > 0 \\ C\mathrm{exp}(Qz) + D\mathrm{exp}(-Qz) & z < 0 \end{cases} \tag{2.10}$$

Due to δ-function at the origin, there is a discontinuity in $f'(z)$ resulting from integrating 2.8 in small vicinity of origin, though the function itself is continuous at the origin. Combining these and demanding that function decays very far away we find

$$\begin{aligned} &A = 0 \qquad D = 0 \qquad \text{decaying far away} \\ &C = B \qquad \text{continuity at origin} \\ &-BQ - CQ = -4\pi\sigma \qquad \Rightarrow \qquad B = \frac{2\pi\sigma}{Q} \qquad \text{discontinuity of } f' \end{aligned} \tag{2.11}$$

Thus, the function f can be written as following

$$f(z) = \frac{2\pi\sigma}{Q}e^{-Q|z|} \tag{2.12}$$

Finally we can write the solution for the vector potential as

$$\mathbf{A} = \mathrm{Re}(\frac{2\pi\sigma}{Q} e^{-Q|z|} e^{-i\omega t}) \cos kx \ \hat{\mathbf{y}} = \frac{2\pi\sigma}{Q} e^{-Q|z|} \cos\omega t \cos kx \ \hat{\mathbf{y}} \tag{2.13}$$

Having the vector potential, it is easy to find electric field $\mathbf{E}$ and magentic field $\mathbf{B}$:

$$\begin{aligned}
\mathbf{E} &= -\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{2\pi\sigma\omega}{Qc} e^{-Q|z|} \sin\omega t \cos kx \ \hat{\mathbf{y}} \\
\mathbf{B} &= \nabla \times \mathbf{A} = \partial_x A_y \hat{\mathbf{z}} - \partial_z A_y \hat{\mathbf{x}} \\
&= \frac{2\pi\sigma}{Q} e^{-Q|z|} \cos(\omega t) \left[-k \sin kx \hat{\mathbf{z}} + Q \ \mathrm{sgn(z)} \cos kx \hat{\mathbf{x}}\right]
\end{aligned} \tag{2.14}$$

In which

$$\mathrm{sgn(z)} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{2.15}$$

(b) $k \leq \omega/c$:

Steps are very similar to case that $k \geq \omega/c$. In this case, general solution for f is given by

$$f(z) = Ae^{iqz} + Be^{-iqz} \tag{2.16}$$

where $q = \sqrt{\frac{\omega^2}{c^2} - k^2}$. Again, by separating solution for positive and negative z, we have

$$f(z) = \begin{cases} Ae^{iqz} + Be^{-iqz} & z > 0 \\ Ce^{iqz} + De^{-iqz} & z < 0 \end{cases} \tag{2.17}$$

We're imposing the condition that $B = C = 0$. This is the same condition as demanding all radiation is outgoing from the sheet of current through infinity. In other words, we're picking only retarded solutions and setting radiation coming from infinity to zero.
The relation between A and D is determined again by continuity of f and discontinuity of f'.

$$\begin{aligned}
&A = D \qquad \text{continuity of } f(z) \\
&A(iq) - D(-iq) = -4\pi\sigma \qquad \text{discontinuity of } f(z) \\
&A = D = \frac{2\pi i\sigma}{q} \\
&f(z) = \frac{2\pi i\sigma}{q} e^{iq|z|}
\end{aligned} \tag{2.18}$$

Thus, the vector potential is given by

$$\mathbf{A} = \text{ Re}\left(\frac{2\pi i\sigma}{q}e^{iq|z|}e^{-i\omega t}\right)\cos kx\ \mathbf{y} = \frac{2\pi\sigma}{q}\sin(\omega t - q|z|)\cos kx\ \hat{\mathbf{y}} \tag{2.19}$$

Therefore electric field and magnetic fields are given by

$$\begin{aligned}
&\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\frac{2\pi\sigma\omega}{qc}\cos(\omega t - q|z|)\cos kx\ \hat{\mathbf{y}} \\
&\mathbf{B} = \nabla\times\mathbf{A} = \partial_x A_y\hat{\mathbf{z}} - \partial_z A_y\hat{\mathbf{x}} \\
&= \frac{2\pi\sigma}{q}\left[-k\sin kx\sin(\omega t - q|z|)\hat{\mathbf{z}} + q\cos kx\ \text{sgn}(z)\ \cos(\omega t - q|z|)\hat{\mathbf{x}}\right]
\end{aligned} \tag{2.20}$$

3. Calculate the time-averaged power radiated by the sheet per unit area of the sheet.

Power per area is given by Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \tag{2.21}$$

$$dP = 2\mathbf{S}\cdot(dA\hat{\mathbf{z}}) \tag{2.22}$$

where factor of 2 is due to two side of the sheet.

(a) $k > \omega/c$:

$$\frac{dP}{dA} = \frac{c}{4\pi}\frac{(2\pi\sigma)^2\omega}{Q^2 c}e^{-2Q|z|}\sin\omega t\cos\omega t\cos kx\left[-k\sin kx\ \hat{\mathbf{x}} - Q\cos kx\ \hat{\mathbf{z}}\right] \tag{2.23}$$

Thus it's easy to see the time averaged power radiated by the sheet is zero:

$$\langle\frac{dP}{dA}\rangle_t = \langle\sin\omega t\cos\omega t\rangle_t\frac{c}{4\pi}\frac{(2\pi\sigma)^2\omega}{Q^2 c}e^{-2Q|z|}\cos kx\left[-k\sin kx\ \hat{\mathbf{x}} - Q\cos kx\ \hat{\mathbf{z}}\right] = 0$$

$$\text{since}\qquad \langle\sin\omega t\cos\omega t\rangle_t = 0 \tag{2.24}$$

This should be intuitive since going to very far away, there is no electric field and magnetic field and that means power radiated to infinity should be on average zero.

(b) $k < \omega/c$:

$$\mathbf{S} = -\frac{c}{4\pi}\frac{(2\pi\sigma)^2\omega}{q^2 c}\cos(\omega t - q|z|)\cos kx$$
$$\times\left[-k\sin kx\sin(\omega t - q|z|)\hat{\mathbf{x}} - q\cos kx\,\mathrm{sgn(z)}\cos(\omega t - q|z|)\hat{\mathbf{z}}\right] \tag{2.25}$$

$$\frac{dP}{dA} = -\frac{2\pi\sigma^2\omega}{\omega^2/c^2 - k^2}\cos(q|z| - \omega t)\cos kx\left[-q\cos kx\ \ \cos(q|z| - \omega t)\right]$$
$$= \frac{2\pi\sigma^2\omega}{\sqrt{\omega^2/c^2 - k^2}}\cos^2(q|z| - \omega t)\cos^2 kx \tag{2.26}$$
$$\langle\frac{dP}{dA}\rangle_t = \frac{2\pi\sigma^2\omega)}{\sqrt{\omega^2/c^2 - k^2}}\cos^2(kx)\langle\cos^2(q|z| - \omega t)\rangle_t = \frac{\pi\sigma^2(\omega/c)}{\sqrt{\omega^2/c^2 - k^2}}\cos^2(kx) \tag{2.27}$$

If we further take average spatially over x-direction, we find:

$$\langle\frac{dP}{dA}\rangle_{t,x} = \frac{\pi\sigma^2\omega}{2\sqrt{\omega^2/c^2 - k^2}} \tag{2.28}$$

where I used the following relation:

$$\langle\cos^2\omega t\rangle_t = \frac{1}{T}\int_0^\infty dt\cos^2(\omega t) = \frac{1}{2} \tag{2.29}$$

where $T = 2\pi/\omega$ is the period of oscillation.