# Physics 6561 Fall 2017 <br> Problem Set 4 Solutions 

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### 4.1 Electrostatics in a rectangular wave-guide

A grounded conducting waveguide has rectangular cross-section in the $x-y$ plane and infinite extent in $z$ :


The $x$ and $y$ dimensions of the waveguide are respectively $a$ and $b$.

1. Find the electrostatic Green's function for this problem, $G\left(x, y, z ; x^{\prime}, y^{\prime}, z^{\prime}\right)$, where (most generally) $0 \leq x \leq a, 0 \leq y \leq b$ and $0 \leq x^{\prime} \leq a, 0 \leq y^{\prime} \leq b$. The boundary condition on $G$ is that it vanishes when the field point $(x, y, z)$ lies on the waveguide.

## Solution due to Dylan Cromer

We want to construct the Green's function for the (negative) Laplace operator with the boundary conditions set by this system. We require the Green's function to vanish at the $x$ and $y$ boundaries of the waveguide. As mentioned in lecture, if we have a linear differential operator $\mathcal{D}$, whise eigenfunctions are $u_{n}$ (each satisfying the boundary conditions) and eigenspectrum $\lambda_{n}$, we can construct the Green's function by

$$
\begin{equation*}
G\left(x, x^{\prime}\right)=\sum_{n} \frac{u_{n}(x) u_{n}^{\dagger}\left(x^{\prime}\right)}{\lambda_{n}} \tag{1.1}
\end{equation*}
$$

This follows from the completeness of the $\mathcal{D}$ eigenbasis. To construct the Green's function in this problem, we will make use of our knowledge of the spectrum of the Laplacian. From the boundary conditions in $x$, our eigenfunctions $u$ will have $x$ dependence given by:

$$
\sqrt{\frac{2}{a}} \sin \left(\frac{n \pi x}{a}\right)
$$

and similarly for $y$ :

$$
\sqrt{\frac{2}{b}} \sin \left(\frac{m \pi y}{b}\right)
$$

Since the $z$ direction is unrestricted, we require a continuous wavenumber $k$ :

$$
e^{i k z}
$$

From this, the entire eigenfunction will be

$$
\begin{equation*}
\frac{2}{\sqrt{a b}} \sin \left(q_{1} x\right) \sin \left(q_{2} y\right) e^{i k z} \tag{1.2}
\end{equation*}
$$

Where $q_{1}=\frac{n \pi}{a}$ and $q_{2}=\frac{\pi m}{b}$. Thus the eigenfunctions must be labeled by $n, m$ and $k$. We will then have two sums and an integral constituting the Green's function. Note that the negative Laplacian, $-\partial_{x}^{2}-\partial_{y}^{2}-\partial_{z}^{2}$ will produce eigenvalues $k^{2}+q_{1}^{2}+q_{2}^{2}$. Finally, the Green's function is given by

$$
\begin{equation*}
G\left(\vec{x}, \vec{x}^{\prime}\right)=\sum_{m, n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{d k}{2 \pi} \frac{4}{a b} \frac{\sin \left(q_{1} x\right) \sin \left(q_{1} x^{\prime}\right) \sin \left(q_{2} y\right) \sin \left(q_{2} y^{\prime}\right) e^{i k z} e^{-i k z^{\prime}}}{k^{2}+q_{1}^{2}+q_{2}^{2}} \tag{1.3}
\end{equation*}
$$

We can rewrite this a bit more schematically, giving

$$
\begin{equation*}
G\left(\vec{x}, \vec{x}^{\prime}\right)=\frac{4}{a b} \sum_{n, m} \sin \left(q_{1} x\right) \sin \left(q_{1} x^{\prime}\right) \sin \left(q_{2} y\right) \sin \left(q_{2} y^{\prime}\right) \int \frac{d k}{2 \pi} \frac{e^{i k\left(z-z^{\prime}\right)}}{k^{2}+q_{1}^{2}+q_{2}^{2}} \tag{1.4}
\end{equation*}
$$

The integral over $k$ is susceptible to complex contour integration, so let's focus on that for now. We can rewrite it as

$$
\begin{equation*}
\int \frac{d k}{2 \pi} \frac{e^{i k(\Delta z)}}{k^{2}+Q^{2}} \tag{1.5}
\end{equation*}
$$

where $Q^{2}=q_{1}^{2}+q_{2}^{2}$ and $\Delta z=z-z^{\prime}$. If we expand $k^{2}+Q^{2}=(k-i Q)(k+i Q)$, we see there are two poles, one at $k=i Q$ and another at $k=-i Q$. For $\Delta z>0$, we want to close the contour in the positive imaginary half-plane to keep the integral regular, and we get

$$
\begin{equation*}
\int \frac{d k}{2 \pi} \frac{1}{k-i Q} \frac{e^{i k \Delta z}}{k+i Q}=\frac{2 \pi i}{2 \pi}\left(\frac{e^{i(-i Q \Delta z)}}{i Q+i Q}\right)=\frac{1}{2} \frac{e^{Q \Delta z}}{Q} \tag{1.6}
\end{equation*}
$$

Thus the result for any $\Delta z$ is

$$
\begin{equation*}
\frac{1}{2} \frac{e^{-Q|\Delta z|}}{Q}=\frac{1}{2} \frac{e^{-\sqrt{\left(\frac{n \pi}{a}\right)^{2}+\left(\frac{m \pi}{b}\right)^{2}}\left|z-z^{\prime}\right|}}{\sqrt{\left(\frac{n \pi}{a}\right)^{2}+\left(\frac{m \pi}{b}\right)^{2}}} \tag{1.7}
\end{equation*}
$$

Putting this together, we get the Green's function as two infinite sums:

$$
\begin{equation*}
G\left(\vec{x}, \vec{x}^{\prime}\right)=\frac{2}{a b} \sum_{n, m} \sin \left(\frac{n \pi x}{a}\right) \sin \left(\frac{n \pi x^{\prime}}{a}\right) \sin \left(\frac{m \pi y}{b}\right) \sin \left(\frac{m \pi y^{\prime}}{b}\right) \frac{e^{-\sqrt{\left(\frac{n \pi}{a}\right)^{2}+\left(\frac{m \pi}{b}\right)^{2}}\left|z-z^{\prime}\right|}}{\sqrt{\left(\frac{n \pi}{a}\right)^{2}+\left(\frac{m \pi}{b}\right)^{2}}} \tag{1.8}
\end{equation*}
$$

2. What is the leading asymptotic behavior of the potential at $(x, y, z)$ due to a point charge $q$ at $\left(x^{\prime}, y^{\prime}, 0\right)$ in the limit $|z| \rightarrow \infty$ ?

If we put a point-charge at $\vec{x}^{\prime}=\left(x^{\prime}, y^{\prime}, 0\right)$, then the potential obeys this Poisson's equation:

$$
\begin{equation*}
-\nabla^{2} \Phi(\vec{x})=4 \pi q \delta^{3}\left(\vec{x}-\vec{x}^{\prime}\right) . \tag{1.9}
\end{equation*}
$$

So when we convolve $G$ with the density to obtain a solution, we will be integrating over a $\delta$ function and simply replace the variable $\vec{x}^{\prime}$ coordinate with the fixed one at the location of the point charge, and multiply by $q$. Thus, the potential is given by

$$
\begin{equation*}
\Phi(\vec{x})=\frac{8 \pi q}{a b} \sum_{n, m} \sin \left(\frac{n \pi x}{a}\right) \sin \left(\frac{n \pi x^{\prime}}{a}\right) \sin \left(\frac{m \pi y}{b}\right) \sin \left(\frac{m \pi y^{\prime}}{b}\right) \frac{e^{-\sqrt{\left(\frac{n \pi}{a}\right)^{2}+\left(\frac{m \pi}{b}\right)^{2}}\left|z-z^{\prime}\right|}}{\sqrt{\left(\frac{n \pi}{a}\right)^{2}+\left(\frac{m \pi}{b}\right)^{2}}} \tag{1.10}
\end{equation*}
$$

To approximate this value at $z \gg 1$, we (as in the lecture) note that the exponential is

$$
\begin{equation*}
\exp \left(-\pi \sqrt{\frac{n^{2}}{a^{2}}+\frac{m^{2}}{b^{2}}}|z|\right) \tag{1.11}
\end{equation*}
$$

This exponentially decays for large $|z|$, and decays slowest for the $n, m=1$ term of the sums. Further terms in the sum decay even faster, so we can approximate the potential at large $|z|$ by only taking the first term of the sum:

$$
\begin{equation*}
\Phi(\vec{x})=\frac{8 q}{\sqrt{a^{2}+b^{2}}} \sin \left(\frac{\pi x}{a}\right) \sin \left(\frac{\pi x^{\prime}}{a}\right) \sin \left(\frac{\pi y}{b}\right) \sin \left(\frac{\pi y^{\prime}}{b}\right) e^{-\frac{\pi}{a b} \sqrt{a^{2}+b^{2}}|z|} \tag{1.12}
\end{equation*}
$$

### 4.2 Spheres in D-dimensions

In D-dimensional spherical coordinates the volume element is written as

$$
\begin{equation*}
d^{D} x=\left(r^{D-1} d r\right)\left(d^{D-1} \Omega_{D-1}\right), \tag{2.13}
\end{equation*}
$$

where the last factor is the ( $D-1$ )-dimensional colume element on the unit sphere in $D$ dimensions.

1. Make a schematic drawing to illustrate the following recursion relation on spherical volume elements:

$$
\begin{equation*}
d^{D} \Omega_{D}=\left(\sin \theta_{D-1}\right)^{D-1} d \theta_{D-1}\left(d^{D-1} \Omega_{D-1}\right) \tag{2.14}
\end{equation*}
$$

As it is shown in figure 1 , the area corresponding to $d^{D} \Omega_{D}$ is equal to :

$$
\begin{align*}
& d S_{D}=R^{D} d^{D} \Omega_{D}=(R d \theta) d S_{D-1}=(R d \theta)\left((R \sin \theta)^{D-1} d^{D-1} \Omega_{D-1}\right) \\
& =R^{D} \sin ^{D-1} \theta d \theta d^{D-1} \Omega_{D-1} \\
& \Rightarrow \quad d^{D} \Omega_{D}=\left(\sin \theta_{D-1}\right)^{D-1} d \theta_{D-1}\left(d^{D-1} \Omega_{D-1}\right) \tag{2.15}
\end{align*}
$$



Figure 1
2. How is $d^{1} \Omega_{1}$ more commonly written?

The lowest dimensional unit sphere is 1 -sphere, a unit circle. So, the volume element is just measuring angles on a unit circle:

$$
\begin{equation*}
d^{1} \Omega_{1}=d \phi \tag{2.16}
\end{equation*}
$$

In which $\phi \in(0,2 \pi]$.
3. Check that the recursion relation correctly gives $d^{2} \Omega_{2}$ and calculate the surface area of the unit sphere in four dimensions by

$$
\begin{equation*}
A_{4}=\int d^{3} \Omega_{3} \tag{2.17}
\end{equation*}
$$

Using 2.15, we have

$$
\begin{equation*}
d^{2} \Omega_{2}=\sin \theta d \theta d \phi \tag{2.18}
\end{equation*}
$$

Which is in fact familiar area element in spherical coordinates for 2 -sphere.
Surface area of the unit sphere in four dimension is given by

$$
\begin{equation*}
A_{4}=\int d^{3} \Omega_{3}=\int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{2 \pi} \sin ^{2} \theta_{2} \sin \theta_{1} d \theta_{2} d \theta_{1} d \phi=2 \pi^{2} \tag{2.19}
\end{equation*}
$$

## A comment about domain of integration

A unit $D$-sphere is defined by following equation in $\mathrm{D}+1$ dimension $\quad \vec{X}=$ $\left(X_{1}, X_{2}, \cdots, X_{D+1}\right) \in \mathbb{R}^{D+1}$.

$$
\begin{equation*}
X_{1}^{2}+X_{2}^{2}+\cdots X_{D+1}^{2}=1 \tag{2.20}
\end{equation*}
$$

knowing parametrization of 2 -sphere, we can generalize it to $D$-sphere:

$$
\begin{align*}
& X_{D+1}=\cos \theta_{D-1} \quad X_{D}=\sin \theta_{D-1} \cos \theta_{D-2} \\
& X_{D-1}=\sin \theta_{D-1} \sin \theta_{D-2} \cos \theta_{D-3} \quad \cdots \quad X_{2}=\sin \theta_{D-1} \sin \theta_{D-2} \cdots \sin \theta_{1} \cos \phi \\
& \quad X_{1}=\sin \theta_{D-1} \sin \theta_{D-2} \cdots \sin \theta_{1} \sin \phi \tag{2.21}
\end{align*}
$$

One can check this indeed solves the defining equation for the $D$-sphere. We now want to determine the range of variables $\left(\theta_{D-1}, \theta_{D-2}, \cdots, \theta_{1}, \phi\right)$. We didn't call that last variables $\theta_{0}$ since its range is different from other coordinates. All other variables $\theta_{i} \in(0, \pi)$, while $\phi \in(0,2 \pi]$. The situation is reminiscent of 2 -sphere in which $\theta \in(0, \pi), \phi \in(0,2 \pi]$. The basic reason is this range of variables are enough to cover sphere once in a smooth way (except at poles). The reason that we don't have to consider $\theta_{i} \geq \pi$ is the following:

$$
\begin{align*}
& X\left(\theta_{D-1}, \theta_{D-2}, \cdots, \theta_{1}+\pi, \phi\right)=X\left(\theta_{D-1}, \theta_{D-2}, \cdots, \pi-\theta_{1}, \phi+\pi\right) \\
& X\left(\theta_{D-1}, \theta_{D-2}, \cdots, \theta_{2}+\pi, \theta_{1}, \phi\right)=X\left(\theta_{D-1}, \theta_{D-2}, \cdots, \pi-\theta_{2}, \pi-\theta_{1}, \phi+\pi\right) \\
& \cdots \quad \cdots  \tag{2.22}\\
& X\left(\theta_{D-1}+\pi, \theta_{D-2}, \cdots, \theta_{2}, \theta_{1}, \phi\right)=X\left(\pi-\theta_{D-1}, \pi-\theta_{D-2}, \cdots, \pi-\theta_{2}, \pi-\theta_{1}, \phi+\pi\right)
\end{align*}
$$

In which $X\left(\theta_{D-1}, \theta_{D-2}, \cdots, \theta_{1}, \phi\right)$ is determining a point on sphere parametrized by $\left(\theta_{D-1}, \theta_{D-2}, \cdots, \theta_{1}, \phi\right)$. This shows, like 2 -sphere, when $\theta_{i} \geq \pi$, we can get to the same point by changing other angles, in particular $\phi \rightarrow \phi+\pi$ and this range of variables is enough to cover the whole sphere.
4. Obtain a general formula for $A_{D}$ by evaluating the integral

$$
\begin{equation*}
\int d^{D} x e^{-x \cdot x} \tag{2.23}
\end{equation*}
$$

first in spherical coordinates and then in Cartesian coordinates and comparing the results. Useful fact:

$$
\begin{equation*}
\int_{0}^{\infty} t^{z-1} e^{-t} d t=\Gamma(z) \quad \text { (gamma function) } \tag{2.24}
\end{equation*}
$$

Let's try to evaluate the following integral

$$
\begin{equation*}
\int d^{D} x e^{-x \cdot x}=\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d x_{1} d x_{2} \cdots d x_{D} e^{-x_{1}^{2}-x_{2}^{2}-\cdots x_{D}^{2}} \tag{2.25}
\end{equation*}
$$

knowing the answer for Gaussian integral $\int_{-\infty}^{+\infty} d x e^{-x^{2}}=\sqrt{\pi}$, we find:

$$
\begin{equation*}
\int d^{D} x e^{-x \cdot x}=\pi^{D / 2} \tag{2.26}
\end{equation*}
$$

On the other hand, we can evaulate the same integral in spherical coordinates:

$$
\begin{align*}
& \int d^{D} x e^{-x \cdot x}=\int_{0}^{+\infty} r^{D-1} d r \int d \Omega_{D-1} e^{-r^{2}}=A_{D} \int_{0}^{+\infty} d r r^{D-1} e^{-r^{2}}= \\
& A_{D} \int_{0}^{+\infty} e^{-t} t^{\frac{D-1}{2}}\left(\frac{d t}{2 t^{1 / 2}}\right)=\frac{A_{D}}{2} \int_{0}^{+\infty} t^{D / 2-1} e^{-t} d t=\frac{A_{D}}{2} \Gamma\left(\frac{D}{2}\right) \\
& A_{D}=\frac{2 \pi^{D / 2}}{\Gamma(D / 2)} \tag{2.27}
\end{align*}
$$

In which I used change of variable $t \equiv r^{2}$. Thus, the area for (D-1)-dimensional unit sphere is given by

$$
\begin{equation*}
A_{D}=\frac{2 \pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \tag{2.28}
\end{equation*}
$$

5. A ball of radius $1 / 2$ fits snugly in a $1 \times 1 \times \cdots \times 1$ box in 10 dimensions. What fraction of the box is empty?

The volume of box is equal to 1 . So, in order to find the fraction that is empty in the box, we need to find volume of 10 -dimensional sphere with radius $1 / 2$ :

$$
\begin{align*}
& V_{D}=\int_{0}^{R} A_{D} r^{D-1} d r=A_{D} \frac{R^{D}}{D} \\
& \Rightarrow \quad V_{10}=\frac{1}{10}\left(\frac{1}{2}\right)^{10} \quad \frac{2 \pi^{5}}{\Gamma(5)}=\frac{\pi^{5}}{122880} \sim 2.49 \times 10^{-3} . \tag{2.29}
\end{align*}
$$

That means the portion of the box which is empty equals to

$$
\begin{equation*}
1-V=0.99651 \tag{2.30}
\end{equation*}
$$

