# Physics 6561 Fall 2017 <br> Problem Set 3 Solutions 

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### 2.1 Radial Laplacian in D-dimensions

Write the radial Laplacian in $D$-dimensions in a form where its self-adjoint property (hermiticity)

$$
\int f(r)\left(\nabla^{2} g(r)\right)\left(r^{D-1} d r\right)=\int\left(\nabla^{2} f(r)\right) g(r)\left(r^{D-1} d r\right)
$$

is very transparent. Henceforth this will be your faviorite way to write the radial Laplacian. Check that the point charge potential, in $D$-dimensions, we used without proof in lecture satisfies Laplace's equation for $r>0$.

The radial Laplacian in $D$-dimensions is given by

$$
\begin{equation*}
\nabla^{2} f=\frac{1}{r^{D-1}} \partial_{r}\left(r^{D-1} \partial_{r} f\right)=\frac{D-1}{r} \frac{\partial f}{\partial r}+\frac{\partial^{2} f}{\partial r^{2}} \tag{1.1}
\end{equation*}
$$

Writing the Laplacian in this form, self-adjointness of the radial Laplacian amounts to usual integration by parts for acting differential on the other function:

$$
\begin{aligned}
& \int f(r)\left(\nabla^{2} g(r)\right)\left(r^{D-1}\right) d r=\int f \partial_{r}\left(r^{D-1} \partial_{r} g\right)= \\
& -\int \partial_{r} f \partial_{r} g r^{D-1} d r=\int \partial_{r}\left(r^{D-1} \partial_{f}\right) g d r=\int\left(\nabla^{2} f(r)\right) g(r)\left(r^{D-1} d r\right)
\end{aligned}
$$

In the above computation, I ignored the boundary terms when I did integration by part. You can think about this as conditions on functions $f, g$ that need to satisfy.

Using this form, it is easy to verify that a point-charge potential is satisfying the radial Laplace equation (except at the origin which there is a delta-function):

$$
\begin{align*}
& V(r)=\frac{1}{r^{D-2}} \\
& \nabla^{2} V=\frac{1}{r^{D-1}} \partial_{r}\left(r^{D-1}\left(-\frac{D-2}{r^{D-1}}\right)\right)=\frac{1}{r^{D-1}} \partial_{r}(-D+2)=0 \tag{1.2}
\end{align*}
$$

2.2Electrostatic energy of a uniformly charges, quadrupolar deformed, fluid drop

This is a variant of the calculation done in lecture (Rayleigh's instability) where instead of all the charge being on the surface (conducting fluid), the fluid has a fixed uniform charge density. The point of the exercise is learning to work with an expansion in a small parameter - the amplitude of the deformation, $\epsilon$.

The drop has radius

$$
r(\theta)=a+\epsilon P_{2}(\cos \theta)+\mathcal{O}\left(\epsilon^{2}\right)
$$

and is comprised of fluid with charge density $\rho_{0}$. Your taski is to show that

$$
\delta U=-\frac{3}{25} \frac{Q^{2}}{a}\left(\frac{\epsilon}{a}\right)^{2}+\cdots
$$

is the change in the electrostatic energy to leading order in the deformation. Here $Q=4 \pi a^{3} \rho_{0} / 3$ is the net charge. Note that this expression is the same as the conducting fluid case with the replacement $1 / 10 \rightarrow 3 / 25$.
We strongly recommend you to do this problem in two steps. Step 1 looks at the longrange effects on the electrostatic energy, where the change to the charge distribution is approximated by a modulated surface charge distribution applied to a spherical drop. In step 2 you calculate the local energy change when the surface charge density of step 1 is converted to a uniform density, a thin layer either added to, or subtracted from, the spherical surface.

1. To a spherical drop of radius $a$ apply the surface charge density

$$
\sigma(\theta)=(r(\theta)-a) \rho_{0}=\epsilon \rho_{0} P_{2}(\cos \theta)
$$

Find the corresponding change $\delta \Phi$ to the potential and compute the resulting energy change $\delta U_{1}$.
2. Show that when the surface charge density of step 1 is converted to an added or subtracted (depending on the sign of $\sigma$ ) layer of charge density $\rho_{0}$, the resulting change in the electric field is confined to a thin layer. Calculate the change in the energy using the energy density $E^{2} / 8 \pi$. Show that this local energy change is always negative and can be written as the following integral over the drop surface:

$$
\delta U_{2}=-\frac{Q}{2 \rho_{0} a^{2}} \int \sigma^{2}(\theta) d A
$$

3. Evaluate $\delta U_{2}$ and obtain the total energy change $\delta U=\delta U_{1}+\delta U_{2}$.

## 1. First solution:

(a) Step 1

For a surface charge denisty given by $\sigma(\theta)=\epsilon \rho_{0} P_{2}(\cos \theta)$, we'll find the potential. Note that this is the charge density obtained shown in figure 1. In order to do so, we will use the expansion of the potential in spherical
coordinates. As usual we separate the regions to inside and outside and use different functions for each region.

$$
\Phi(\vec{r})= \begin{cases}\Phi_{\text {in }}(\vec{r})=\sum_{l=0}^{\infty} A_{l} r^{l} P_{l}(\cos \theta) & r \leq R  \tag{2.1}\\ \Phi_{\text {out }}(\vec{r})=\sum_{l=0}^{\infty} B_{l} r^{-l-1} P_{l}(\cos \theta) & r \geq R\end{cases}
$$

We know that potential should be continuous and it's derivative on two side, proportional to electric field, should produce the charge density $\sigma(\theta)$.
Continuity implies


Figure 1: $\delta \rho$ in space, found by the overlap between original sphere and squished sphere. Volume is constant, so $\delta \rho(\vec{r})$ contains both positive and negative charges.

$$
\begin{equation*}
\sum_{l=0}^{\infty} B_{l} a^{-l-1} P_{l}(\cos \theta)=\sum_{l=0}^{\infty} A_{l} a^{l} P_{l}(\cos \theta) \quad \Rightarrow \quad B_{l}=A_{l} a^{2 l+1} \tag{2.2}
\end{equation*}
$$

Now applying discontinuity of electric field, we find:

$$
\begin{align*}
& \partial_{r} \Phi_{\text {out }}-\partial_{r} \Phi_{\text {in }}=-4 \pi \sigma \\
& \sum_{l=0}^{\infty}(-l-1) B_{l} a^{-l-2} P_{l}(\cos \theta)-\sum_{l=0}^{\infty} l A_{l} a^{l-1} P_{l}(\cos \theta)=-4 \pi \sigma \\
& \sum_{l=0}^{\infty}(2 l+1) A_{l} P_{l}(\cos \theta)=4 \pi \sigma(\theta)=4 \pi \epsilon a P_{2} \cos (\theta) \tag{2.3}
\end{align*}
$$

By using orthogonality of Legendre polynomials $A_{l}=\frac{4 \pi \epsilon \rho_{0}}{5 a} \delta_{l, 2}$. Thus, we find the potential as

$$
\Phi(\vec{r})=\left\{\begin{array}{l}
\Phi_{\text {in }}(\vec{r})=\frac{4 \pi \epsilon \rho_{0}}{5 a} r^{2} P_{2}(\cos \theta)  \tag{2.4}\\
\Phi_{\text {out }}(\vec{r})=\frac{4 \pi \epsilon \rho_{0} a^{2}}{5} \frac{P_{2}(\cos \theta)}{r^{3}} \quad r \geq R
\end{array}\right.
$$

Using this potential expression, we find energy difference due to this potential:

$$
\begin{align*}
& \delta U_{1}=\frac{1}{2} \int \delta \Phi \delta \rho d V=\frac{2 \pi \epsilon \rho_{0} a}{5} \int \rho_{0} P_{2}(\cos \theta) \sigma(\theta) d A \\
& \frac{2 \pi \epsilon^{2} \rho_{0}^{2} a^{3}}{5} \int_{0}^{\pi} \int_{0}^{2 \pi} P_{2}(\cos \theta)^{2} d \phi \sin \theta d \theta=\frac{8 \pi^{2} \epsilon^{2} \rho_{0}^{2}}{25} \\
& =\frac{8 \pi^{2} \epsilon^{2} a^{3}}{25}\left(\frac{3 Q}{4 \pi a^{3}}\right)^{2}=\frac{9}{50} \frac{Q^{2}}{a}\left(\frac{\epsilon}{a}\right)^{2} \tag{2.5}
\end{align*}
$$

## (b) Step 2



Figure 2: Using superposition for computing the potential energy. The squished sphere can be thought as sum of the right hand side charge configurations.

From the hint is problem, one can infer that you're asked to solve the problem as it is shown in figure 2. The energy is given by
$U=\frac{1}{8 \pi} \int \vec{E} \cdot \vec{E} d V=\frac{1}{8 \pi} \int \vec{E}_{1} \cdot \vec{E}_{1} d V+\frac{1}{4 \pi} \int \vec{E}_{1} \cdot \vec{E}_{2} d V+\frac{1}{8 \pi} \int \vec{E}_{2} \cdot \vec{E}_{2} d V$

Where integrals are done over all the space. What we did in step 1 , was really computing the first term, self-energy of $\sigma$ with itself. Here is a trickey part: when you consider superposition of second charges in figure 2, the result for electric field is the electric field due to perfect sphere, and $\delta E$ which has support only in small range of $\delta r$. This is a dipole term which is very similar to electric field inside a capacitor. This contribution is shown in figure 3, and using Gauss law $\delta E$ is given by'

$$
\begin{gather*}
\delta E=-4 \pi \rho_{0}(a+\delta(r(\theta))-r)  \tag{2.7}\\
\delta r(\theta) \\
\rho_{0} \\
-\sigma(\theta)
\end{gather*}
$$

Figure 3: zooming into contant angle part of surface charge density $\sigma$ and $\rho_{0}$ stretched from $a$ to $a+\delta(r(\theta))$

Interestingly, we don't need to compute $\vec{E}_{1} \cdot \vec{E}_{2}$ since $\int d V \vec{E}_{1} \cdot \vec{E}_{s}$ is zero (verify this!), and contribution inside the capacitor is third order in $\delta r$. The reason is $E_{1}$ is already first order in $\delta r$, the electric field inside the capacitor is also first order in $\delta r$ and the domain we're integrating is also first order in $\delta r$, so we don't need to compute this cross term $\vec{E}_{1} \cdot \overrightarrow{E_{2}}$.
As a result, we need to only compute $\vec{E}_{2} \cdot \vec{E}_{2}$ :

$$
\begin{align*}
& \int \vec{E}_{2} \cdot \vec{E}_{2} d V=\int \vec{E}_{s} \cdot \vec{E}_{s} d V+2 \int \vec{E}_{s} \cdot(\delta E \delta(a \leq r \leq a+\delta r) \hat{r}) d V+\mathcal{O}(\delta r)^{4}  \tag{2.8}\\
& \Rightarrow \quad \delta U_{2}=-\frac{Q}{a^{2}} \int d A \int_{a}^{a+\delta r(\theta)} \rho_{0}(a+\delta r(\theta)-r) d r \\
& =-\frac{Q \rho_{0}}{a^{2}} \int d A \frac{\delta r(\theta)^{2}}{2}=-\frac{Q}{2 \rho_{0} a^{2}} \int \sigma(\theta)^{2} d A \tag{2.9}
\end{align*}
$$

By computing this integral, we find:

$$
\begin{equation*}
\int \sigma^{2} d A=\left(\epsilon \rho_{0}\right)^{2} \int_{0}^{\pi} \int_{0}^{2 \pi} a^{2} P_{2}(\cos \theta)^{2} \sin \theta d \theta=\frac{4 \pi^{2} \epsilon^{2} \rho_{0}^{2}}{5} \tag{2.10}
\end{equation*}
$$

That yields $\delta U_{2}$,

$$
\begin{equation*}
\delta U_{2}=-\frac{Q}{2 \rho_{0} a} \frac{4 \pi \epsilon^{2} \rho_{0}^{2} a^{2}}{5}=-\frac{3}{10} \frac{Q^{2}}{a}\left(\frac{\epsilon}{a}\right)^{2} \tag{2.11}
\end{equation*}
$$

Sum of two terms yield

$$
\begin{equation*}
\delta U=\delta U_{1}+\delta U_{2}=\left(\frac{9}{50}-\frac{3}{10}\right) \frac{Q^{2}}{a}\left(\frac{\epsilon}{a}\right)^{2}=-\frac{3}{25} \frac{Q^{2}}{a}\left(\frac{\epsilon}{a}\right)^{2} \tag{2.12}
\end{equation*}
$$

## 2. Second solution:

This solution is more mathematical and precise. We'll keep track of all the terms to second order in $\epsilon$. Also I'm keeping $\delta R$ general function of $\theta, \phi$ for most of this solution in order to easily generalize it to one of the application of this question which is about oblateness of Earth.
The potential energy for a given arbitrary charge distribution, denoted by $\rho(r)$ is given by:

$$
\begin{equation*}
U=\frac{1}{2} \int \frac{\rho(\vec{r}) \rho\left(\vec{r}^{\prime}\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} \vec{r} d^{3} \vec{r}^{\prime} \tag{2.13}
\end{equation*}
$$

Domain of integrals are the whole space $\mathbf{R}^{3}$. All the information regarding where the charges are located is implicit inside the function $\rho(\vec{r})$. For a sphere with radius $R$ :

$$
\rho(\vec{r})= \begin{cases}\rho_{0} & |r|<a  \tag{2.14}\\ 0 & |r|>a\end{cases}
$$

Now we're deforming the sphere. That means the function $\rho(\vec{r})$ has changed slightly and we'd like to calculate corresponding change in the potential energy to leading order in $\epsilon$ :

$$
\begin{align*}
\delta U= & \frac{1}{2} \int \frac{\delta \rho(\vec{r}) \delta \rho\left(\vec{r}^{\prime}\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} \vec{r} d^{3} \vec{r}^{\prime}+\frac{1}{2} \int \frac{\delta \rho(\vec{r}) \rho\left(\vec{r}^{\prime}\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} \vec{r} d^{3} \vec{r}^{\prime}+\frac{1}{2} \int \frac{\rho(\vec{r}) \delta \rho\left(\vec{r}^{\prime}\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} \vec{r} d^{3} \vec{r}^{\prime} \\
& =\frac{1}{2} \int \frac{\delta \rho(\vec{r}) \delta \rho\left(\vec{r}^{\prime}\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} \vec{r} d^{3} \vec{r}^{\prime}+\int \frac{\delta \rho(\vec{r}) \rho\left(\vec{r}^{\prime}\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} \vec{r} d^{3} \vec{r}^{\prime}=\delta U_{1}+\delta U_{2} \tag{2.15}
\end{align*}
$$

Where I swapped $\vec{r} \leftrightarrow \vec{r}^{\prime}$ in the third term and hence, second term multiplied by two.
Note above equation is true for any variation of $\delta \rho$, whether it's small or large. When variation is small, we typically expect to keep only the second term. However, here first order term in $\epsilon$ won't change the potential energy which can be seen by either explicit calculation or the expectation that sphere has the minimum electric potential energy when it has uniform density, so we really need to keep track of terms to second order. Now let's write the new density function:

$$
\rho(\vec{r})+\delta \rho(\vec{r})= \begin{cases}\rho_{0} & r<a+\delta R(\hat{r})  \tag{2.16}\\ 0 & r>a+\delta R(\hat{r})\end{cases}
$$

Here I assumed the boundary of object is determined by $r=a+\delta R(\hat{r})$, where $\hat{r}$ is a symbol for $(\theta, \phi)$.
In this problem, we're keeping $\delta R(\hat{r})$ at most to second order in $\epsilon$. To first order
in this probelm, $\delta R=\epsilon P_{2}(\cos \theta)$. To second order, demanding the same volume for new shape of object, we have:

$$
\begin{align*}
& \int d V=\int d \Omega \int_{0}^{a+\delta R} r^{2} d r=\frac{4 \pi a^{3}}{3} \\
& \Rightarrow \quad \int\left(a^{2} \delta R+a \delta R^{2}\right) d \Omega=0 \tag{2.17}
\end{align*}
$$

Where $\int d \Omega=\int_{0}^{\pi} \int_{0}^{2 \pi} \sin \theta d \theta d \phi$ is integral over solid angles. We omitted third order term $\delta R^{3}$ in volume since it is enough to keep the answer to second order.

Let's compute $\delta U_{2} . \delta U_{2}$ can be computed in two different ways:

$$
\begin{equation*}
\delta U_{2}=\int \delta \Phi\left(\vec{r}^{\prime}\right) \rho\left(\vec{r}^{\prime}\right) d^{3} \vec{r}^{\prime}=\int \delta \rho(\vec{r}) \Phi(\vec{r}) d^{3} \vec{r} \tag{2.18}
\end{equation*}
$$

where $\delta \Phi\left(\vec{r}^{\prime}\right)$ is the potential generated by $\delta \rho(\vec{r}) . \Phi(\vec{r})$ is the potential of old charge configuration which was a perfect sphere and given by:

$$
\Phi(r)= \begin{cases}\frac{4 \pi \rho_{0} a^{3}}{3 r} & r \geq a  \tag{2.19}\\ \frac{4 \pi \rho_{0} a^{2}}{3}+\frac{2 \pi \rho_{0} a^{2}}{3}\left(1-\frac{r^{2}}{a^{2}}\right) & r \leq a\end{cases}
$$

I used the second way of computing $\delta U_{1}$, which is done by considering old potential at the location of new charge distribution. The first way should be doable as well. Note that $\delta R(\hat{r})$ is positive and negative depending (look at figure 1) on the solid angle $\hat{r}$ since volume should be conserved. So we need to use different forms of function, written explicitly above as the potential when we're evaluating it on the locations of $\delta \rho(\vec{r})$. Moreover, $\int \delta \rho d^{3} r$ is at least first order in $\delta R$, so when we're trying to compute $\delta U_{2}$ to second order, we need to find the potential at most to first order. Electric field is continuous at $r=R$ and that means we don't really need to seperate domain of integration to $r>a$ and $r<a$ and use different functions for the potential. Thus, we used $\frac{4 \pi a^{3}}{3 r}$ as the potential function, evaluated in the location of new charges $\delta \rho$. This is a tricky point regarding using perturbation theory, so it's a good time to pause and make sure this makes sense to you. Having said that,

$$
\begin{align*}
\delta U_{2}= & \int \delta \rho(\vec{r}) \Phi(\vec{r}) d^{3} \vec{r}=\int_{a}^{a+\delta R(\hat{r})} \rho_{0}\left(\frac{4 \pi \rho_{0} a^{3}}{3 r}\right) r^{2} d r d \Omega \\
& =\frac{4 \pi \rho_{0}^{2} a^{3}}{3} \int \frac{1}{2}\left(\delta R^{2}+2 a \delta R\right) d \Omega \\
& =\frac{2 \pi \rho_{0}^{2} a^{3}}{3} \int\left(\delta R^{2}-2 \delta R^{2}\right) d \Omega \\
& =-\frac{2 \pi \rho_{0}^{2} a^{3}}{3} \int \delta R^{2} d \Omega \tag{2.20}
\end{align*}
$$

where I used (2.17) in above equation. This way of writing it is useful since in the final form, we only need to know $\delta R$ to first order in small parameter $\epsilon$. For this problem, $\delta R=\epsilon P_{2}(\cos \theta)+\mathcal{O}\left(\epsilon^{2}\right)$,

$$
\begin{equation*}
\delta U_{2}=-\frac{2 \pi \rho_{0}^{2} a^{3} \epsilon^{2}}{3} \int\left(P_{2}(\cos \theta)\right)^{2} d \Omega=-\frac{8 \pi^{2} \rho_{0}^{2} a^{3} \epsilon^{2}}{15}=-\frac{3 Q^{2} \epsilon^{2}}{10 a^{3}} \tag{2.21}
\end{equation*}
$$

Now let's evaluate the $\delta U_{1}$. This is equal to work that we need to do in order to bring all $\delta \rho$ charges to their current locations from infinity. In other words, this is self energy of $\delta \rho$. For $\delta U_{1}$, we need to know $\delta \rho$ to first order and it is enough to compute $\frac{1}{\left|\vec{r}-\vec{r}^{\prime}\right|}$ to zero order:

$$
\begin{align*}
& \delta U_{1}=\frac{1}{2} \int \frac{\rho_{0} a^{2} \delta R(\hat{r}) \rho_{0} a^{2} \delta R\left(\hat{r^{\prime}}\right)}{\left|R \hat{r}-R \hat{r}^{\prime}\right|} d \Omega d \Omega^{\prime}= \\
& \frac{\rho_{0}^{2} a^{3}}{2} \int \delta R(\hat{r}) \delta R\left(\hat{r}^{\prime}\right)\left(\sum_{l=0}^{\infty} \frac{4 \pi}{2 l+1} P_{l}\left(\cos \gamma_{\hat{r}, \hat{r}^{\prime}}\right)\right) d \Omega d \Omega^{\prime}= \\
& \frac{\rho_{0}^{2} a^{3}}{2} \int \delta R(\hat{r}) \delta R\left(\hat{r}^{\prime}\right)\left(\sum_{l=0}^{\infty} \frac{4 \pi}{2 l+1} \sum_{m=-l}^{l} Y_{l m}(\hat{r}) Y_{l m}^{*}\left(\hat{r}^{\prime}\right)\right) d \Omega d \Omega^{\prime}= \\
& \frac{\rho_{0}^{2} a^{3}}{2} \sum_{l=0}^{\infty} \frac{4 \pi}{2 l+1} \sum_{m=-l}^{l}\left|\int \delta R(\hat{r}) Y_{l m}(\hat{r}) d \Omega\right|^{2} \\
& \frac{\rho_{0}^{2} a^{3}}{2} \frac{4 \pi}{5} \frac{4 \pi}{5}=\frac{9 Q^{2} \epsilon^{2}}{50 a^{3}} \tag{2.22}
\end{align*}
$$

Where I used $\int P_{l^{\prime}}(\cos \theta) Y_{l m}(\theta, \phi) d \Omega=\delta_{m, 0} \delta_{l, l^{\prime}} \sqrt{\frac{4 \pi}{2 l+1}}$ and spherical harmonic addition theorem.
Combining these, we have:

$$
\begin{equation*}
\delta U=\left(\frac{9}{50}-\frac{3}{10}\right) \frac{Q^{2} \epsilon^{2}}{a^{3}}=-\frac{3 Q^{2}}{25 a}\left(\frac{\epsilon^{2}}{a^{2}}\right) \tag{2.23}
\end{equation*}
$$

3. Application: Oblateness of Earth One application of this problem is to determine how oblate Earth is: When Earth was almost in liquid phase, one can consider it as an almost spherical gravitating object, and gravitational potential energy is obtained by exactly the same technique we used in this problem. So its shape can be inferred by minimizing its potential energy. However, because of Earth's rotation, we need to include the rotational energy as well.
If we expand change in radius in terms of spherical harmonics

$$
\begin{align*}
& \delta R(\hat{r})=\sum_{l, m} a_{l m} Y_{l m}(\theta, \phi)  \tag{2.24}\\
& a_{l m}=\int \delta R(\hat{r}) Y_{l m}^{*}(\theta, \phi) d \Omega \tag{2.25}
\end{align*}
$$

Gravitational potential energy is sum of $\delta U_{1}$ and $\delta U_{2}$ followed as

$$
\begin{equation*}
\delta U_{G}=\frac{2 \pi G \rho_{0}^{2} R^{3}}{3} \sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(1-\frac{3}{2 l+1}\right)\left|a_{l m}\right|^{2} \tag{2.26}
\end{equation*}
$$

$a_{00}$ to first order is zero due to (2.17). Also $a_{1, m}$ 's are not appeared in $\delta U$ since to first order they're corresponding to rigid translations of object and by choosing center of mass reference frame we can set them to zero. As it is written, the change in potential is positive for any $a_{l m}$, so in the absence of potential energy, we should set all to zero. However, centrifugal force makes $a_{20}$ non-zero as we calculate it now. For a particle with mass $\delta m$, moving with angular frequency $\omega$ with distance $s$ from the axis of rotation

$$
\begin{equation*}
\delta U_{R}=-\frac{\delta m \omega^{2} s^{2}}{2} \tag{2.27}
\end{equation*}
$$

Unlike gravitational potential energy that we need to compute it to second order, it is enough to evaluate rotational energy to the first order. Therefore, we have

$$
\begin{align*}
& U_{R}=-\int \frac{\delta \rho(\vec{r}) \omega^{2} r^{2} \sin ^{2} \theta}{2} d^{3} \vec{r}=\frac{\rho_{0} \omega^{2}}{2} \int d \Omega \int_{R}^{R+\delta R(\hat{r})} r^{4} d r \sin ^{2} \theta= \\
& -\frac{\rho_{0} \omega^{2}}{2} \int R^{4} \delta R \sin ^{2} \theta d \Omega= \\
& -\frac{\rho_{0} \omega^{2} R^{4}}{2} \int \delta R(\hat{r}) \sin ^{2} \theta d \Omega=-\frac{\rho_{0} \omega^{2} R^{4}}{2} \int \delta R(\hat{r})\left(\frac{2-2 P_{2}(\cos \theta)}{3}\right) \\
& =\frac{\rho_{0} \omega^{2} R^{4}}{3} \sqrt{\frac{4 \pi}{5}} a_{20} \tag{2.28}
\end{align*}
$$

Where I used $\int \delta R(\hat{r}) d \Omega=0$ to first order. Total energy change is given by

$$
\begin{equation*}
\delta U=\frac{2 \pi G \rho_{0}^{2} R^{3}}{3} \sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(1-\frac{3}{2 l+1}\right)\left|a_{l m}\right|^{2}+\frac{\rho_{0} \omega^{2} R^{4}}{3} \sqrt{\frac{4 \pi}{5}} a_{20} \tag{2.29}
\end{equation*}
$$

Minimizing energy yields

$$
\begin{align*}
& \frac{4 \pi G \rho_{0}^{2} R^{3}}{3}\left(1-\frac{3}{5}\right) a_{20}+\frac{\rho_{0} \omega^{2} R^{4}}{3} \sqrt{\frac{4 \pi}{5}}=0 \\
& a_{20}=-\frac{5}{6} \sqrt{\frac{4 \pi}{5}} \frac{3 \omega^{2} R}{4 \pi \rho_{0} G} \\
& \delta R(\hat{r})=-\frac{5}{8 \pi} \frac{\omega^{2} R}{G \rho_{0}} P_{2}(\cos \theta)=-\frac{5}{6} \frac{\omega^{2} R^{2}}{g} P_{2}(\cos \theta) \tag{2.30}
\end{align*}
$$

In which $g$ is gravity acceleration constant on surface of Earth. Using this result, we find at poles $\theta=0$,

$$
\begin{equation*}
\delta R_{p}=-\frac{5}{6} \frac{\omega^{2} R^{2}}{g} \tag{2.31}
\end{equation*}
$$

and in equator $\theta=\pi / 2$,

$$
\begin{equation*}
\delta R_{e}=\frac{5}{12} \frac{\omega^{2} R^{2}}{g} \tag{2.32}
\end{equation*}
$$

So the difference between radius at poles and equator is equal to

$$
\begin{equation*}
\Delta R=\frac{5}{4} \frac{\omega^{2} R^{2}}{g} \tag{2.33}
\end{equation*}
$$

For Earth,

$$
\begin{equation*}
g=9.8 \mathrm{~m} / \mathrm{s}^{2} \quad R=6.4 \times 10^{6} \mathrm{~m} \quad \omega=\frac{2 \pi}{T}=7.3 \times 10^{-5} \mathrm{~s} \tag{2.34}
\end{equation*}
$$

Where $T$ is period of rotation of Earth ( $\mathrm{T}=24 \mathrm{hr}$ ). Numerically,

$$
\begin{equation*}
\Delta R=28 \mathrm{~km} \tag{2.35}
\end{equation*}
$$

The measured value for this quantity is 21 km .
Note that density of Earth is not really a constant and it increases more toward
the center due to compression. Interestingly enough, by modeling density of Earth radially changing toward the center, and knowing just one more number which is exact moment of inertial of Earth, one can obtain $\Delta R=22 \mathrm{~km}$ which is perfect agreement with observed value. Equator itself is also not a perfect circle, and the difference between longest and shortest radius is about 200 m .

## References

[1] http://mamwad.org/x1/x1-008.pdf
This is the source I used for second solution and application part of the problem. However, it is not written in English.

