Physics 6561 Fall 2017
Problem Set 2 Solutions

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2.1 *Conformally invariant knot energy*

In an effort to find the simplest shape for a given knotted topology of a loop of string, mathematicians have considered minimizing an energy model where all pairs of line elements $d\mathbf{r}_1$ and $d\mathbf{r}_2$ of the string interact 'electrostatically':

$$dU = \frac{|d\mathbf{r_1}|\,|d\mathbf{r_2}|}{|\mathbf{r_1} - \mathbf{r_2}|^2} \tag{1.1}$$

E&M was never their strong suit, so we forgive them the exponent 2 in their Coulomb's law! The total energy has the form of a double integral:

$$U = \oint ds \oint dt \left(\frac{|\dot{\mathbf{r}}(s)|\,|\dot{\mathbf{r}}(t)|}{|\mathbf{r}(s) - \mathbf{r}(t)|^2} - \frac{1}{(s-t)^2} \right). \tag{1.2}$$

The second term in the integrand, which does not depend on the shape of the knot, is needed to make the expression finite. As physicists we admire the fact that this U is invariant with respect to translations and rotations of the knot. And thanks to that exponent 2, we see that it is even invariant with respect to changes in scale. Show that invariance extends to all elements of the group of conformal transformations by verifying it for Kelvin inversions.

We need to verify that the integrand is invariant under inversions:

$$\begin{aligned}
\mathbf{r} \to \mathbf{r}' &= \frac{\mathbf{r}}{\mathbf{r}\cdot\mathbf{r}} \\
\dot{\mathbf{r}}'(t) &= \frac{\dot{\mathbf{r}}(t)}{\mathbf{r}(t)\cdot\mathbf{r}(t)} - \frac{2\mathbf{r}(t)\,[\mathbf{r}(t)\cdot\dot{\mathbf{r}}(t)]}{(\mathbf{r}(t)\cdot\mathbf{r}(t))^2} \\
|\dot{\mathbf{r}}'(t)|^2 &= \frac{\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)}{(\mathbf{r}\cdot\mathbf{r})^2} + \frac{4(\dot{\mathbf{r}}\cdot\mathbf{r})^2}{(\mathbf{r}\cdot\mathbf{r})^3} - \frac{4(\dot{\mathbf{r}}\cdot\mathbf{r})^2}{(\mathbf{r}\cdot\mathbf{r})^3} = \frac{\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)}{(\mathbf{r}\cdot\mathbf{r})^2}
\end{aligned} \tag{1.3}$$

Also for denominator goes as:

$$\mathbf{r}(s) - \mathbf{r}(t) \to \frac{\mathbf{r}(s)}{\mathbf{r}(s)\cdot\mathbf{r}(s)} - \frac{\mathbf{r}(t)}{\mathbf{r}(t)\cdot\mathbf{r}(t)} \tag{1.4}$$

$$\Rightarrow \qquad |\mathbf{r}(s) - \mathbf{r}(t)|^2 \to \frac{1}{\mathbf{r}(s)\cdot\mathbf{r}(s)} + \frac{1}{\mathbf{r}(t)\cdot\mathbf{r}(t)} - \frac{2\mathbf{r}(s)\cdot\mathbf{r}(t)}{\mathbf{r}(t)\cdot\mathbf{r}(t)\mathbf{r}(s)\cdot\mathbf{r}(s)}$$

$$= \frac{(\mathbf{r}(s) - \mathbf{r}(t))^2}{\mathbf{r}(t)\cdot\mathbf{r}(t)\mathbf{r}(s)\cdot\mathbf{r}(s)} \tag{1.5}$$

Note that using this equation, we can find again (1.3) by setting $t \to s + \delta s$ and taking

the limit.
Combining these two result, we find invariance of the total energy:

$$\frac{|\dot{\mathbf{r}}(s)|\,|\dot{\mathbf{r}}(t)|}{|\mathbf{r}(s)-\mathbf{r}(t)|^2} \to \frac{\frac{|\dot{\mathbf{r}}(s)|}{\mathbf{r}^2(s)}\frac{|\dot{\mathbf{r}}(t)|}{\mathbf{r}^2(t)}}{\frac{|\mathbf{r}(s)-\mathbf{r}(t)|^2}{\mathbf{r}^2(t)\mathbf{r}^2(s)}} = \frac{|\dot{\mathbf{r}}(s)|\,|\dot{\mathbf{r}}(t)|}{|\mathbf{r}(s)-\mathbf{r}(t)|^2} \tag{1.6}$$

Which shows invariance under inversion at origin.

2.2 *Contacting conducting spheres*

Two identical conducting spheres of radius r make electrical contact and together carry net charge Q. The spheres are initially at rest. When released, they fly apart in opposite directions, each carrying charge $Q/2$. What value does the net kinetic energy approach as the distance between the spheres grows large? Ignore gravity and energy loss due to radiation and friction. *Hint*: use Kelvin inversion.

We can use energy conservation to find final kinetic energy provided that we know initial potential energy when spheres are in contact with each other. More explicitly,

$$U_0 = \frac{Q^2}{2C_0} = K_{\text{net}} + U_f = K + 2\frac{(Q/2)^2}{2C_{sphere}} \qquad C_{sphere} = r \tag{2.1}$$

In which the fact that final potential energy is the sum of each sphere potential energy is assumed when the distance between them is very large. This means finally we can ignore their interactions. So the problem boils down to finding capacitance of initial charge distribution:

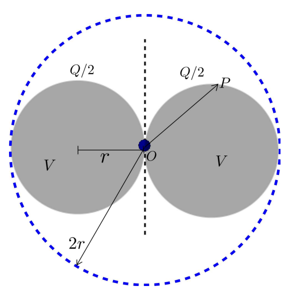


Figure 1: Blue circle is the circle of inversion

We can find capacitance by considering infinite image charges inside each sphere, however, using Kelvin inversion, the geometry of charge distribution will be very simple. The surface of each sphere is mapped to infinite plane, showed in figure 2, and inside each sphere is mapped to inside of infinite sheets, extending to infinity. The region which is outside of spheres is mapped to the white region, inside infinite planes.

Note that inversion won't keep the potential constant, therefore equipotential surfaces or conductors are **not** mapped to equipotential surfaces. More explicitly, if we consider a point P, shown in figure 2 with potential V which is the potential of conductor, is mapped under inversion to the point P', with potential V', then we have

$$OP\,OP' = (2r)^2 \qquad \frac{V'}{V} = \frac{2r}{OP'} = \frac{OP}{2r} \tag{2.2}$$

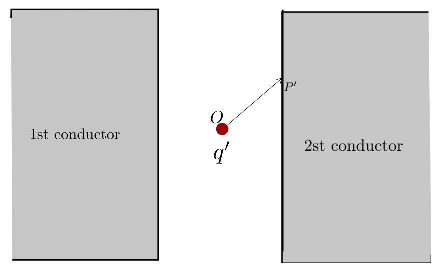


Figure 2: inside conductors are mapped to gray regions. OP' is in the same direction as OP.

This means the potential due to charges on the surface of each plate is just given by $\frac{2Vr}{OP'}$ which is exactly the potential due to a point-charge at the origin. We want to guess what kind of charge distribution on the plates can generate this potential. It's not hard to see that this is just the same charge distribution when we're placing a charge $q' = -2Vr$ in front of two infinite conductors which have zero potential. Because for this latter problem, the potential produced by charges on the plates should cancel potential due to point charge at point P' which is just $\frac{q'}{OP'}$. However, we know how to solve the problem of a point-charge in front of two infinite conducting sheets. This consists of placing infinite image charges with alternating signs as it's shown in figure 3.

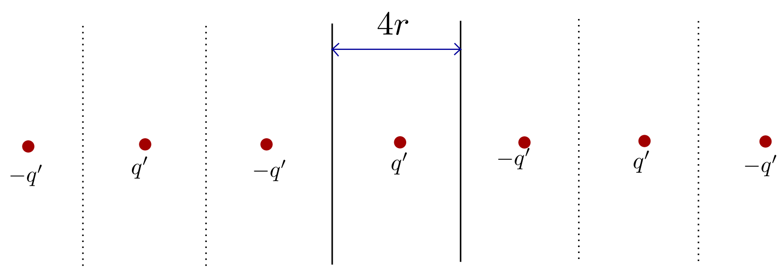


Figure 3

We also know from the example of point charge in front of spheres solved in class, that a point-charge q at point R is mapped to a point charge q' at point R' in the following way:

$$\frac{q'}{q} = -\frac{2r}{R'} = \frac{R}{2r} \tag{2.3}$$

Again be careful about charges when you do Kelvin inversion. There is no relation between **total** charges, however, for each point charge (2.3) holds. Combining these facts, we can find total charge on each capacitor in terms of $q' = 2Vr$:

$$Q/2 = -\frac{2r}{4r}(-q') + \frac{2r}{8r}(-q') - \frac{2r}{12r}(-q') + \cdots \tag{2.4}$$
$$= \frac{q'}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots) = \frac{q'}{2}\log 2 = Vr\log 2 \tag{2.5}$$

This immediately gives the capacitance for the system:

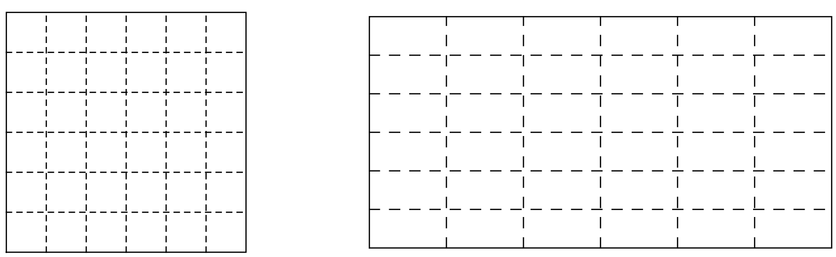
$$C = 2r\log 2 \approx 1.38r \tag{2.6}$$
$$K_{\text{each}} = \frac{Q^2}{2r}(\frac{1-\log 2}{4\log 2}) \tag{2.7}$$

Note that capacitance for the system is bigger than capacitance of a single sphere (should we expect this?), and smaller than capacitance of two identical spheres with the same radius connected with thin wire, when they are far away from each other.

2.3 *Mapping the square, conformally, to the rectangle*

Consider conformal maps that send the interior of a square to the interior of a rectangle, and also preserve vertical and horizontal mirrors.

1. First explain why the following (scaling just the x-coordinate) is *not* conformal:



It's obvious from the figure that if we consider any of diagonal lines with horizontal line, after conformal transformations the angle between these curves won't be 45 and will change, hence this can't be a conformal transformations.

2. Make a sketch of the true conformal map (one that preserves mirrors).

This is a sketch of conformal transformation that can map square to rectangle. Schwarz-Christoffel mapping is telling us how to construct such a map explicitly since Schwarz-Christoffel transformation maps upper half plane onto the interior

of a simple polygon. For the purpose of this question, we should bear in mind mapping is only preserving angles in the **interior** of two regions, but it can change the angle of the boundaries since typically conformal transformations won't be regular at the boundary and theorem regarding preserving the angle by conformal transformation won't hold.

3. What 2D electrostatics problem is solved by this conformal map?

This is a application of this conformal transformation. By examing the picture and figure 4, vertical lines in the square are now starting and ending on an interval smaller than size of the rectangle. So we can think about this interval as parts of system of periodic capacitors extended to infinity. Another way of thinking about this is as we have a fluid and we put periodic blocks ahead of the stream of the fluid.

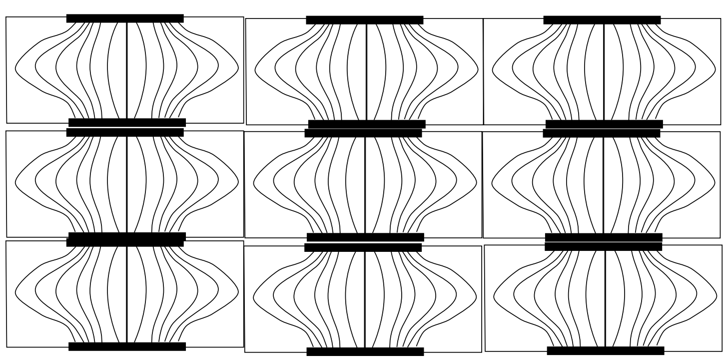


Figure 4: Infinite series of parallel capacitors

2.4*Exercise from The Lost Jackson Codex, Vol. XIV*
Evaluate the following dimensionless double integral

$$\oint_{C_1} \oint_{C_2} \frac{d\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{4.1}$$

for the pair of closed curves C_1 and C_2 shown on the next page. The orientations are indicated by arrows.
Hint: everybody knows that combining things, e.g. chocolate+ peanut-butter, can produce great things. In this case, chocolate= Ampere, peanut-butter = Biot-Savart.

The double integral

$$\oint_{C_1} \oint_{C_2} \frac{d\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{4.2}$$

Can be thought as integrating magnetic field, due to unit current $I = 1$ flowing in the second loop, over the first loop. More explicitly, for a unit current the magnetic field is given by:

$$\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \oint_{C_2} \frac{d\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{4.3}$$

Then, Ampere's law tells us:

$$\oint_{C_1} \mathbf{B}(\mathbf{r}_1) \cdot d\mathbf{r}_1 = \int_{D_1} \nabla \times \mathbf{B} \cdot d\mathbf{s} = \int_{D_1} \mu_0 \mathbf{J}(r_1) \cdot d\mathbf{s} = -\mu_0 \int_{D_1} J ds = -\mu_0 \tag{4.4}$$

Where D_1 is the interior of the region having C_1 as its boundary. There is a minus sign since the current density J and area element $d\mathbf{s}$ are in opposite direction when we used right-hand rule for both determining positive cross product and defining positive area element.
Hence,

$$\oint_{C_1}\oint_{C_2} \frac{d\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \frac{4\pi}{\mu_0} \oint_{C_1} \mathbf{B}(\mathbf{r}_1) \cdot d\mathbf{r}_1 = -4\pi \tag{4.5}$$