Physics 6561 Fall 2017
Problem Set 1 Solutions

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1.1 *Quadrupole moment of the charged conducting disk*

Using the potential Φ from lecture for the charged conducting disk, show that for large r (distance from the disk center)

$$\Phi(r,\theta) \sim \frac{Q}{r} + A_2 \frac{P_2(\cos(\theta))}{r^3} + \cdots \tag{1.1}$$

As we found in class, the potential Φ for a charged conducting disk is given by:

$$\Phi(\psi) = \frac{Q}{2}\int_{\psi}^{\infty} \frac{d\psi'}{(a^2+\psi')\sqrt{\psi'}} \tag{1.2}$$

Since we'd like to find large distance expansion of the potential, we can start expanding (1.2) in large ψ. Another approach is by exactly computing the integral and then expanding for large r which can be done for above integral. Although in this example both methods are accessible, but generally the first method has the advantage of simplifying the integrand and then integrating.

1.1 1st Solution:

In this approach, let's find the potential approximately, to leading order and subleading order in terms of ψ. Also, let's solve ψ equation perturbatively in terms of $\frac{a}{r}$.

$$\Phi = \frac{Q}{2}\int_{\psi}^{\infty} \frac{d\psi'}{\psi^{3/2}}\left[1 - \frac{a^2}{\psi'} + \mathcal{O}\left(\frac{a^4}{\psi'^2}\right)\right] = \tag{1.3}$$

$$\Rightarrow \qquad \Phi = Q\left(\frac{1}{\sqrt{\psi}} - \frac{1}{3}\frac{a^2}{\psi^{3/2}}\cdots\right) \tag{1.4}$$

$$\frac{x^2}{a^2+\psi} + \frac{y^2}{b^2+\psi} + \frac{z^2}{c^2+\psi} = 1 \tag{1.5}$$

For disk, $a = b$, and $c = 0$. In terms of spherical coordinates,

$$
\begin{aligned}
z &= r\cos\theta \\
y &= r\sin\theta\sin\phi \\
x &= r\sin\theta\cos\phi
\end{aligned} \tag{1.6}
$$

$$\frac{x^2+y^2}{a^2+\psi} + \frac{z^2}{\psi} = 1 \tag{1.7}$$

$$\frac{r^2\sin^2\theta}{a^2+\psi} + \frac{r^2\cos^2\theta}{\psi} = 1 \tag{1.8}$$

$$\frac{r^2\sin^2\theta}{\psi}(1-\frac{a^2}{\psi}+\cdots) + \frac{r^2\cos^2\theta}{\psi} = 1 \tag{1.9}$$

$$\text{Ignoring } \frac{a^2}{\psi} \;\Rightarrow \psi = r^2 \tag{1.10}$$

$$\text{next to leading term} \qquad \psi = r^2(1+\delta\psi) \qquad \Rightarrow \delta\psi = -\frac{a^2}{r^2}\sin^2\theta \tag{1.11}$$

$$\psi = r^2(1-\frac{a^2}{r^2}\sin^2\theta) \tag{1.12}$$

Combining (1.7) and (1.3), one recovers the form of the potential in the problem:

$$\Phi = Q\left(\frac{1}{\sqrt{r^2(1-\frac{a^2}{r^2}\sin^2\theta)}} - \frac{1}{3}\frac{a^2}{r^3}\right) = Q\left(\frac{1}{r}(1+\frac{a^2}{2r^2}\sin^2\theta) - \frac{1}{3}\frac{a^2}{r^3}\right) \tag{1.13}$$

$$= \frac{Q}{r} - \frac{Qa^2}{3r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) \tag{1.14}$$

$$A_2 = \frac{-Qa^2}{3} \tag{1.15}$$

1.2 2nd Solution:

By changing variable to $\psi' = a^2\tan^2(x)$, one finds:

$$\Phi(\psi) = \frac{Q}{a}\left(\frac{\pi}{2} - \arctan\left(\frac{\sqrt{\psi}}{a}\right)\right) \tag{1.16}$$

This has the correct large distance behaviour since when $\psi \to \infty$, arctan $\to \pi/2$,and$\Phi \to 0$.

$$\begin{aligned} &\frac{x^2+y^2}{a^2+\psi}+\frac{z^2}{\psi}=1 \\ \Rightarrow \quad & \psi(\psi+a^2)-\psi(x^2+y^2)-(\psi+a^2)z^2=0 \\ \Rightarrow \quad & \psi=\frac{x^2+y^2+z^2-a^2\pm\sqrt{(x^2+y^2+z^2-a^2)^2+4a^2z^2}}{2} \end{aligned} \tag{1.17}$$

Since potential (1.16) is given in terms of $\sqrt{\psi}$, in order to get a real number we need to pick the positive root of square root. Finally using change of coordinates in terms of spherical coordinates,

we find $\Phi(r,\theta)$:

$$\Phi(r,\theta)=\frac{Q}{a}(\frac{\pi}{2}-\arctan\left(\sqrt{\frac{r^2-a^2+\sqrt{(r^2-a^2)^2+4a^2r^2\cos^2\theta}}{2a^2}}\right) \tag{1.18}$$

If you're using Mathematica, using Series[$\cdots$, $\{r,\infty,2\}$] we can find large r dependence. Expanding potential in powers of r yields

$$\Phi(r,\theta)\approx\frac{Q}{r}-\frac{Qa^2}{3}\frac{P_2(\cos\theta)}{r^3}+\frac{Qa^4}{5}\frac{P_4(\cos\theta)}{r^5}+\mathcal{O}(\frac{a}{r})^7 \tag{1.19}$$

Note for showing that the potential has this form, we need to keep θ arbitrary in order to see whether this function can be written as $\sum_l(B_lr^l+C_lr^{-l-1})P_l(\cos\theta)$. In other words, let's assume we are given a function of $\Phi(r,\theta)$ and someone **claims** this function is the potential of a charge distribution. Then because we know potential should obey Laplace equation, once we're expanding the function at large distances, there should be delicate relations between powers of $\frac{1}{r^n}$ and Legendre polynomials $P_n(\cos\theta)$. For instance the function multiplied by $1/r^3$ should be merely $P_2(\cos\theta)$. Having said that, if we know **for sure** that the given function is the potential function of some charge distribution, then maybe quickest way to find A_2 is to expand the potential along the z direction.

1.2 *Charge distribution on the charged conducting disk*

As another application of the potential Φ, obtain the surface charge density $\sigma(r)$, where r is the distance from the center of the disk.

Gauss law tells us surface charge density for a conductor is given by

$$\sigma=-\frac{1}{4\pi}\frac{\partial\Phi}{\partial n} \tag{2.1}$$

where $\mathbf{n}$ is the normal direction at that point.

So if you think about disk as an ellipsoid when we sent $c\to 0$, then disk has two side and for each side, the surface charge density can be found using above equation.

$$\sigma(r)^+ = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial z}|_{z=\varepsilon} = \frac{1}{4\pi}E_z(z=\varepsilon) \tag{2.2}$$

$$\sigma(r)^- = \frac{1}{4\pi}\frac{\partial\Phi}{\partial z}|_{z=-\varepsilon} = -\frac{1}{4\pi}E_z(z=-\varepsilon) \tag{2.3}$$

Where ε is potivie and we're taking it $\varepsilon \to 0$.

$$\begin{aligned} E_z(x,y,z) =& \frac{\sqrt{2}Qz}{\sqrt{4a^2z^2+(x^2+y^2+z^2-a^2)^2}} \\ &\times \frac{1}{\sqrt{-a^2+z^2+x^2+y^2+\sqrt{4a^2z^2+(x^2+y^2+z^2-a^2)^2}}} \end{aligned} \tag{2.4}$$

Now if we expand this expression close to the surface $z=0$,

$$E_z(x,y,\varepsilon) \approx \frac{\sqrt{2}Q\varepsilon}{(a^2-x^2-y^2)}\frac{1}{\sqrt{\frac{2a^2\varepsilon^2}{a^2-x^2-y^2}}} \tag{2.5}$$

$$\Rightarrow E_z(x,y,\varepsilon) \approx \frac{Q}{\sqrt{a^2-x^2-y^2}}\frac{\varepsilon}{|\varepsilon|} \tag{2.6}$$

$$\lim_{\varepsilon\to 0} E(x,y,\varepsilon) = \frac{Q}{\sqrt{a^2-x^2-y^2}} \tag{2.7}$$

$$\lim_{\varepsilon\to 0} E(x,y,-\varepsilon) = -\frac{Q}{\sqrt{a^2-x^2-y^2}} \tag{2.8}$$

$$\Rightarrow \qquad \sigma^+(x,y) = \frac{Q}{4\pi\sqrt{a^2-x^2-y^2}} \tag{2.9}$$

$$\Rightarrow \qquad \sigma^-(x,y) = \frac{Q}{4\pi\sqrt{a^2-x^2-y^2}} \tag{2.10}$$

Note that for obtaining (2.5) from (2.4), we used $x^2+y^2 \leq a^2$ mutiple times for simplifying square roots.

Using this, we can verify the total charge on disk indeed is given by Q, which is integral of sum of the surface charge densities $\sigma = \sigma^+ + \sigma^-$ over the disk:

$$\int_{x^2+y^2\leq a^2} \sigma(x,y)dxdy = \int_0^a \rho d\rho \int_0^{2\pi} d\phi \frac{Q}{2\pi\sqrt{a^2-\rho^2}} = Q \tag{2.11}$$

1.3 *Exercise from the Lost Jackson Codex, Vol. XIV*

Suppose $\mathbf{r}(s)$, $s \in [0,1]$, is a smooth closed curve in the $x-y$ plane. What are the possible values of the following double integral?

$$\mathcal{J} = \int_0^1 ds \int_{s^+}^1 dt\ \hat{\mathbf{z}} \cdot \dot{\mathbf{r}}(s) \times \dot{\mathbf{r}}(t)\delta^2(\mathbf{r}(s)-\mathbf{r}(t)) \tag{3.1}$$

The last factor in the integrand is standard shorthand for the product of two Dirac deltas:

$$\delta^2(\mathbf{r}(s) - \mathbf{r}(t)) = \delta(x(s) - x(t))\delta(y(s) - y(t)) \tag{3.2}$$

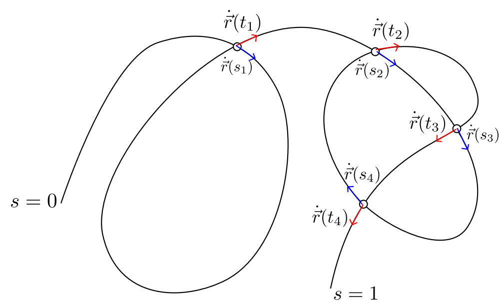

Figure 1: An example of a curve yielding non-zero value for $\mathcal{J}$

Delta function forces us to evaluate integrand when $\mathbf{r}(s) = \mathbf{r}(t)$. However integrand $\hat{\mathbf{z}} \cdot \dot{\mathbf{r}}(s) \times \dot{\mathbf{r}}(t)$ yields zero when $\hat{\dot{\mathbf{r}}}(\mathbf{s}) = \hat{\dot{\mathbf{r}}}(\mathbf{t})$, so if we allow $t = s$ the integrand is zero while delta function is non-zero and the integral is not well-defined. Note that this scenario was excluded in the second edition of problem set due to range of t, $t \in (s, 1]$. Thus, the only possibility for finding a non-zero answer is to have points that are intersecting but velocities are not pointing along the same direction, as shown below.
Let's evaluate integral for these curves. One subtlety is δ-functions are functions of $x(s) - x(t)$, $y(s) - y(t)$, so we need to take into account Jacobian of transformations between s, t and $x(s), x(t)$ (similarly for $y(t), y(s)$). To be more concrete, let's digress for a moment and try to evaluate following integral. First, let's find the one-dimensional integral:

$$\int dx \delta(f(x)) g(x) \tag{3.3}$$

$$f(x) = 0 \qquad \Rightarrow \qquad x = x_0 \tag{3.4}$$
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = f'(x_0)(x - x_0) \tag{3.5}$$
$$\int dx \delta(f'(x_0)(x - x_0)) g(x) = \int dx \frac{\delta(x - x_0)}{|f'(x_0)|} g(x) = \frac{g(x_0)}{|f(x_0)|} \tag{3.6}$$

Now let's consider a two-dimensional example.

$$\int ds dt \delta\left(f_1(s,t)\right) \delta\left(f_2(s,t)\right) g(s,t) \tag{3.7}$$

Let's assume $f_1(s_0, t_0) = 0$, $f_2(s_0, t_0) = 0$, and there is a one-to-one relationships between

(s,t) and (f_1, f_2) in the vicinity of (s_0, t_0). Then, using change of variables theorem,

$$\int dsdt\delta\left(f_1(s,t)\right)\delta\left(f_2(s,t)\right)g(s,t) \tag{3.8}$$

$$= \int df_1 df_2 \frac{1}{\begin{vmatrix}\partial_s f_1 & \partial_t f_1\\ \partial_s f_2 & \partial_t f_2\end{vmatrix}} \delta(f_1)\delta(f_2)g(s,t) = \tag{3.9}$$

$$\frac{g(s,t)}{\begin{vmatrix}\partial_s f_1 & \partial_t f_1\\ \partial_s f_2 & \partial_t f_2\end{vmatrix}}|_{(s_0,t_0)} = \frac{g(s_0,t_0)}{|\partial_s f_1(s_0,t_0)\partial_t f_2(s_0,t_0) - \partial_t f_1(s_0,t_0)\partial_s f_1(s_0,t_0)|} \tag{3.10}$$

Now let's apply this method and its generalization to our problem. I'm going to assume the curve has N intersection point and the curve is passing only twice through each intersection point. Therefore, in the vicinity of each intersection points, we can repeat above argument ($f_1 = x(s) - x(t), f_2 = y(s) - y(t)$):

$$\hat{\mathbf{z}} \cdot \dot{\mathbf{r}}(s) \times \dot{\mathbf{r}}(t) = \dot{x}(s)\dot{y}(t) - \dot{y}(s)\dot{x}(t) \tag{3.11}$$

$$\int_0^1 ds \int_{s^+}^1 dt\left(\dot{x}(s)\dot{y}(t) - \dot{y}(s)\dot{x}(t)\right)\delta(x(s) - x(t))\delta(y(s) - y(t)) = \tag{3.12}$$

$$\sum_{i=1}^{N} \frac{(\dot{x}(s_i)\dot{y}(t_i) - \dot{y}(s_i)\dot{x}(t_i))}{|(\dot{x}(s_i)\dot{y}(t_i) - \dot{y}(s_i)\dot{x}(t_i))|} = \sum_{i=1}^{N} \mathrm{sign}(\sin(\theta_i)) \tag{3.13}$$

Where θ_i is the angle between $\vec{\mathbf{r}}(s)$ and $\vec{\mathbf{r}}(t)$ and

$$\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{3.14}$$

For example, this integer number is $+1 + 1 - 1 + 1 = 2$ for the drawn curve.