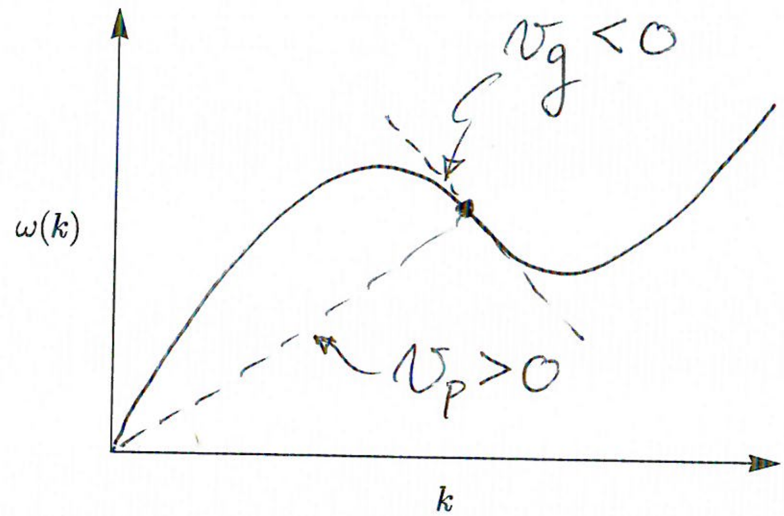
1. *(15 points)* Dispersion relations are usually plotted for just positive $k$ :




The convention is that $k > 0$ is a right-moving wave. There are also left-moving waves with $k < 0$ that have the same frequency, or $\omega(-k) = \omega(k)$.

In the plot above sketch the dispersion relation for a hypothetical medium where the phase and group velocities of wavepackets can have **opposite signs**.

2. *(15 points)* A "classical" particle moving in the plane has momentum

$$p_x = p \qquad\qquad p_y = 0$$

(a) Write down the two components of the central wavevector when the particle is described by a Schrödinger wavepacket.

$$k_x = \frac{p}{\hbar} \qquad\qquad k_y = 0$$

(b) The 2D Schrödinger wavepacket has equal size in the two dimensions:

$$\delta x = r \qquad\qquad \delta y = r$$

Estimate the corresponding uncertainties in its momentum:

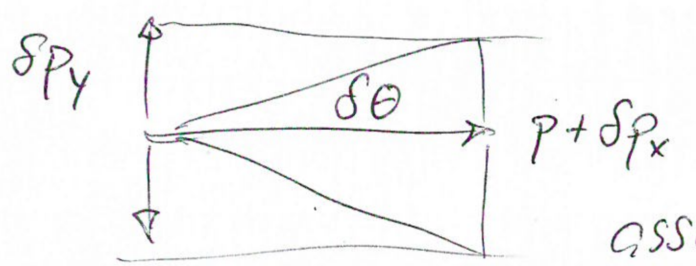
$$\delta p_x = \frac{\hbar}{r} \qquad\qquad \delta p_y = \frac{\hbar}{r}$$

(c) Because of its wave nature, the particle is **not** precisely moving in the $x$-direction (as classical mechanics has us believe). Estimate the range of angles in its momentum direction, $\delta\theta$.

$$\hbar \vec{k} = \vec{p}$$

$$\delta x \, \delta p_x \sim \hbar$$
$$\delta y \, \delta p_y \sim \hbar$$

$$\delta\theta \sim \frac{\delta p_y}{p} = \frac{\hbar}{pr}$$




assume $\delta p_x \ll p$
so still close to
"classical"

3. *(15 points)* A sinusoidal sound wave (in air, from a distant source) moves in the $+x$ direction and has wavelength $\lambda$.

   (a) Write down the molecule displacements in the wave, $\mathbf{s}(x,t)$. Your expression should be a **vector** field and include $\lambda$. Any other parameters should be briefly defined in words.

$$\mathbf{s}(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x-v t)\right)\hat{x}$$

$A$ = amplitude, $v$ = speed of sound

   (b) Suppose the wave has maximum density modulation $\delta n_{\max}/n = 10^{-3}$ and wavelength $\lambda = 3.14\,\mathrm{m}$. Use the relation

$$\frac{\delta n}{n} = -\nabla\cdot\mathbf{s}$$

   to find the maximum displacement (relative to equilibrium) of the air molecules, $|\mathbf{s}|_{\max}$.

$$\vec{\nabla}\cdot\vec{S} = A\cdot\frac{2\pi}{\lambda}\cos(\cdots)$$

$$\left|-\vec{\nabla}\cdot\vec{S}\right|_{max} = 2\pi\frac{A}{\lambda} = 10^{-3}$$

$$|\vec{S}|_{max} = A = 10^{-3}\frac{\lambda}{2\pi} = 10^{-3}(.5m) = .5\,mm$$

4. *(15 points)* Write down the most general expression for a **left-moving** wave in a one-dimensional medium with dispersion relation $\omega(k)$:

$$\psi(x,t) = \int_0^\infty dk\, A(k) \cos\left(kx + \omega(k)t + \phi(k)\right)$$

5. *(15 points)* A wave in a one-dimensional medium with dispersion relation $\omega(k)$ can be described as a function of position, $\psi(x,t)$, or alternatively (using the Fourier transform) as a function of wavevector, $\hat{\psi}(k,t)$.

   Briefly describe the dynamics of $\hat{\psi}(k,t)$, emphasizing how it is **much simpler** than the dynamics of $\psi(x,t)$.

$$\hat{\psi}(k,t) = \underbrace{\hat{\psi}(k,0)}_{\text{set by initial conditions}} e^{-i\omega(k)t}$$

At each $k$ the phasor $\hat{\psi}(k,t)$ just rotates at constant angular velocity $\omega(k)$.

6. *(10 points)* In a sound wave the density and average velocity of the particles in the gas oscillate in both position and time. By the process called **thermalization**, two statistical properties are hypothesized to eventually be restored over time.

   (a) Briefly describe these properties:

   1. positions are uniformly distributed
   2. velocities are isotropically distributed

   (b) What mechanical property[1] of the equilibrium gas did we derive that relied on the two hypotheses?

$E(L)$ or $E(V)$ with $V = A \cdot L$

Volume dependence of mechanical energy.

[1] Detailed formula not required!

7. *(15 points)* In Prelim 1 the electric field of an electromagnetic wave in one frame,

$$E_y = f(x - ct)\ ,$$

was found to have the form

$$E'_y = f(r\,(x' - ct'))\ ,$$

in a frame moving with velocity $u$ in the same direction as the electromagnetic wave, where $r$ is the rescaling factor

$$r = \sqrt{\frac{1 - u/c}{1 + u/c}}\ .$$

(a) Write down a function $f$ that corresponds to a wave with wavelength $\lambda$.

$$f(y) = A\sin\left(\frac{2\pi}{\lambda}y\right)$$

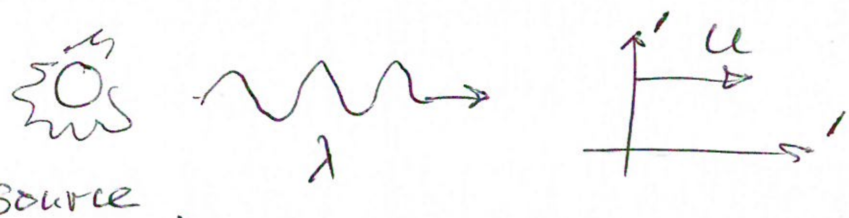
(b) What is the ratio of wavelengths in the two frames,

$$E_y' = A\sin\left(\underbrace{\frac{2\pi}{\lambda}r}_{}(x' - ct')\right)$$

$\lambda'/\lambda = \dfrac{1}{r}$

$$\frac{2\pi}{\lambda'} \Rightarrow \lambda' = \frac{\lambda}{r}$$

(c) Evaluate $\lambda'/\lambda =$ for the case that $\lambda$ is the wavelength in the rest frame of the source, and the primed frame is moving **away** from the source at 80% of the speed of light.




Source (at rest)

$$\lambda' \cancel{/} \lambda = \frac{1}{r} = \left(\frac{1+.8}{1-.8}\right)^{1/2}$$

$$= \left(\frac{1.8}{.2}\right)^{1/2} = \sqrt{9} = 3$$