1. *(15 points)* Recall the following two approximations that were made in the derivation of the water-surface-wave dispersion relation:

$$\dot{s}(x,t) = v_z(x, s(x,t)) \approx v_z(x,0)$$

$$K = \int_A dx dy \int_{-\infty}^{s(x,t)} dz \, \left(\frac{1}{2}\rho v^2\right) \approx \int_A dx dy \int_{-\infty}^{0} dz \, \left(\frac{1}{2}\rho v^2\right) .$$

Write down the dimensionless quantity that was assumed to be small when making these approximations.

$s(x,t)/\lambda$ or $s(x,t) \cdot k$

2. *(20 points)* Consider an infinite elastic string whose small-amplitude vertical (y) motion is described by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} .$$

Suppose the entire string has been at rest for all times before $t = 0$. Starting at time $t = 0$ a mechanism moves the point $x = 0$ of the string up and down in some fashion described by the function $h(t)$, where $h(t) = 0$ for $t < 0$.

(a) Write down a formula for the amplitude of the string $y(x, t)$ for $t > 0$ in terms of the function h but just in the region $x > 0$.

There cannot be any left-moving wave since going back any amount of time it will still be in $x>0$, just further to the right:

$$y(x,t) = f(x-vt) + 0 = h(t - x/v)$$

$$h(t) = y(0,t) = f(\underbrace{0-vt}_{y}) \Rightarrow f(y) = h(-y/v)$$

(b) Repeat what you just did, but in the region $x < 0$.

In this region there can only be a left-moving wave:

$$y(x,t) = 0 + g(x+vt)$$

$$h(t) = y(0,t) = g(\underbrace{vt}_{y}) \Rightarrow g(y) = h(y/v)$$

$$\Rightarrow y(x,t) = h(t + x/v)$$

3. *(20 points)* The air pressure[1] (p) in an organ pipe satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = v^2 \frac{\partial^2 p}{\partial x^2} \, ,$$

where x is the coordinate along the length of the organ pipe and v is the speed of sound in air. One end of the pipe is closed, the other is open. At the open end, $x = 0$, the pressure satisfies

$$p(0, t) = 0$$

at all times t. At the closed end, $x = L$, the pressure satisfies

$$\frac{\partial p}{\partial x}(L, t) = 0$$

at all times t. The two boundary conditions limit the possible **normal-mode** oscillation frequencies of the pressure in the pipe. Work out the lowest three frequencies ω, expressing your answers in terms of v and L.

$p(x,t) = A \cos\omega t \sin kx$, so $p(0,t) = 0$

wave equation: $-A\omega^2 \cos\omega t \sin kx = -v^2 k^2 \cos\omega t \sin kx$

$\omega = vk$

$\frac{\partial p}{\partial x} = kA \cos\omega t \cos kx$, $\cos kL = 0 \Rightarrow k = \frac{\pi}{L} \times \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}$

$\omega = \pi \frac{v}{L} \times \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right\}$

[1]In this problem p is actually the *change* in pressure relative to the equilibrium atmospheric pressure.

4. *(20 points)* Here is the Schrödinger equation for a particle of mass m moving in a one-dimensional potential $U(x)$:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi\ . \tag{1}$$

Here is a modification of the Schrödinger equation, again for a particle of mass m, but where the potential has been replaced with the squared-magnitude of Ψ:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + |\Psi|^2\Psi\ . \tag{2}$$

(a) Give one property that equation (1) has but (2) does not.

linearity

(b) Give one property that equation (2) has but (1) does not.

space-translation symmetry
$(x \longrightarrow x + x_0)$

5. *(25 points)* As we saw in lecture, there are electromagnetic "plane waves" where the electric field has the form $\mathbf{E} = E_y(x,t)\,\hat{\mathbf{y}}$, and the function E_y satisfies the wave equation:

$$\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2} .$$

moving to right (pos. x)

Consider the solution

$$E_y(x,t) = f(x - ct) , \tag{3}$$

where f is some function of one argument. The same electromagnetic wave seen in a moving frame, with field E'_y, is related to the original one by the equation

$$E'_y(x',t') = E_y(x,t) ,$$

where the primed coordinates refer to the moving frame. Because the form of the wave equation is unchanged by a Lorentz transformation, you are not surprised that

$$E'_y(x',t') = g(x' - ct') ,$$

where g is some other function of one variable.

Use the Lorentz transformation to find how f and g are related. The moving frame is moving with speed u in the same direction as the electromagnetic wave is moving.

(a) Since this is the last problem, we are going to guide you. Start by writing down the Lorentz transformation of coordinates. Express the unprimed coordinates in terms of the primed coordinates, and remember the primed frame is moving in the same direction as the electromagnetic wave.

$x = \gamma(x' + ut')$

$t = \gamma(t' + \frac{u}{c^2}x')$

the point $x' = 0$ is moving to right in unprimed frame (x increasing)

(b) Substitute your formulas in (a) into the argument of f in (3).

$$E'_y(x',t') = E_y(x,t) = f\Big(\underbrace{\gamma(x'+ut') - c\gamma(t' + \tfrac{u}{c^2}x')}_{\gamma(1-\frac{u}{c})(x'-ct')}\Big)$$

(c) Now tidy up your answer for (b) so the relationship between f and the function g in the primed frame is obvious.

$$E_y' = f\left(\gamma\left(1-\frac{u}{c}\right)(x'-ct')\right)$$
$$= g(x'-ct')$$

$$\Rightarrow g(y) = f\left(\gamma\left(1-\frac{u}{c}\right)y\right)$$

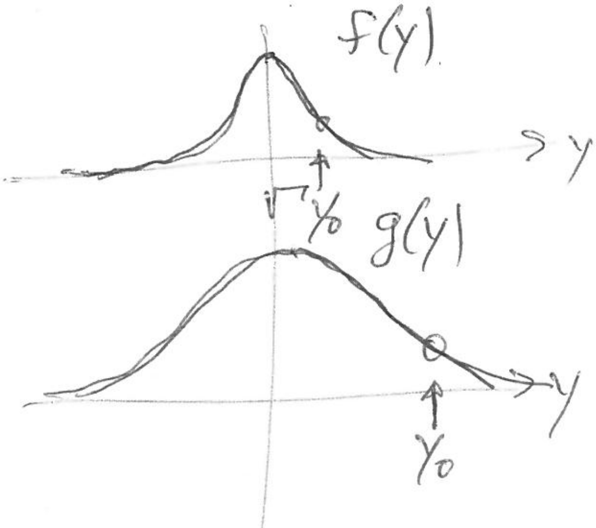
simple scaling applied to argument

scaling factor $= \gamma\left(1-\frac{u}{c}\right)$

$$= \frac{1}{(1-u^2/c^2)^{1/2}}\left(\left(1-\frac{u}{c}\right)^2\right)^{1/2}$$

$$= \left(\frac{\cancel{(1-u/c)}(1-u/c)}{\cancel{(1-u/c)}(1+u/c)}\right)^{1/2}$$

$$= \sqrt{\frac{1-\beta}{1+\beta}}, \quad \beta = u/c$$



$$g(y_0) = f\left(\sqrt{\frac{1-\beta}{1+\beta}}\, y_0\right)$$