origins

- splitting method for PDEs
  Douglas-Rachford (DR) splitting, 1956

- convex optimization
  alternating direction method of multipliers (ADMM), 1974

- phase retrieval
  Fienup’s hybrid input-output (HIO), 1982

Douglas-Rachford splitting

Douglas, Jim, Jr., and Henry H. Rachford Jr.

"On the numerical solution of heat conduction problems in two and three space variables."

Transactions of the American mathematical Society

82.2 (1956): 421-439. **(1452 citations)**

Idea:

"Split" equations into two sets, one involving only x-derivatives, the other only y-derivatives, and devise an iteration scheme that solves the 2D problem where in each step you only need to solve 1D problems.

https://regularize.wordpress.com/2017/02/24/the-origin-of-the-douglas-rachford-iteration/

(Dirk Lorenz site)

ADMM

Also motivated by numerical solution of PDEs
(Dirichlet problem for the s-Laplacian operator)
R. Glowinski & A. Marrocco

S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein

"Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers"

Foundations and Trends in Machine Learning, 3(1):1–122, 2011.

(6541 citations)

online:
https://web.stanford.edu/~boyd/papers/admm_distr_stats.html

ADMM variables

$$p'_1 = P_1(p_2 - u)$$
$$u' = u + \alpha(p'_1 - p_2)$$
$$p'_2 = P_2(p'_1 + u)$$

$$u = \text{“accumulated constraint discrepancy”}$$

$$\text{define } x = p_2 - u, \quad \text{set } \alpha = 1:$$

$$x' = x + P_2(2P_1(x) - x) - P_1(x)$$

result: RRR with $\beta = 1$

hybrid input-output algorithm (HIO)

Fienup, James R.
"Phase retrieval algorithms: a comparison."
Applied optics 21.15 (1982): 2758-2769.

(4067 citations)

Still another method of choosing the next input which was investigated is a combination of the upper line of Eq. (43) with the lower line of Eq. (42):

$$g_{k+1}(x) = \begin{cases} g'_k(x), & x \notin \gamma, \\ g_k(x) - \beta g'_k(x), & x \varepsilon \gamma. \end{cases} \tag{44}$$

$g(x)$: Fourier magnitude projection of x

$x \notin \gamma$: $x \in$ support

difference map

Elser, Veit
“Phase retrieval by iterated projections.”
JOSA A 20.1 (2003): 40-55.

$$x' = x + \beta \left(P_2((1+\gamma)P_1(x) - \gamma x) - P_1((1-\gamma)P_2(x) + \gamma x)\right)$$

$$\gamma = 1/\beta$$

“derived” from simple Ansatz and local fixed-point analysis

https://en.wikipedia.org/wiki/Difference-map_algorithm

Jonathan Borwein
(1951-2016)

reflect-reflect-average: $$x' = \frac{x + R_2(R_1(x))}{2}$$

$$\text{reflect in constraint } i: \quad R_i(x) = 2P_i(x) - x$$

relaxed-reflect-reflect = RRR :

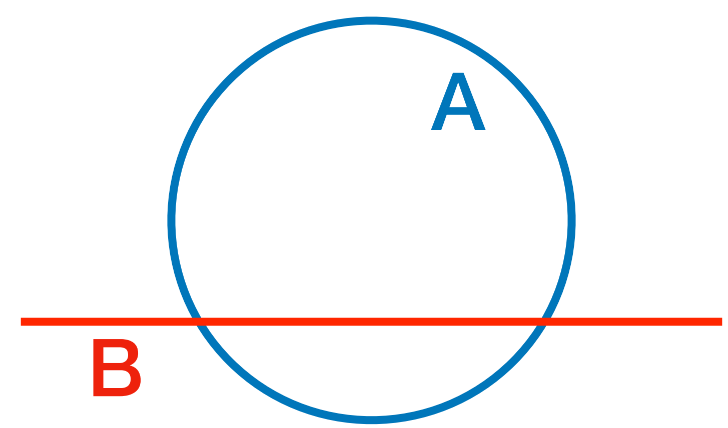
$$x' = (1 - \beta/2)x + (\beta/2)R_2(R_1(x))$$

$$x' = x + \beta\,(P_2(2P_1(x) - x) - P_1(x))$$

Dear Veit.
Do send my best to Steve.
In general for projection and reflection methods, especially in the convex case, **under-relaxation can improve convergence theory** while slowing things down.

I would be interested in knowing more about your examples where it is helping?

Cheers, Jon



Artacho, Francisco J. Aragón, and Jonathan M. Borwein.
"Global convergence of a non-convex Douglas–Rachford iteration."
Journal of Global Optimization 57.3 (2013): 753-769.