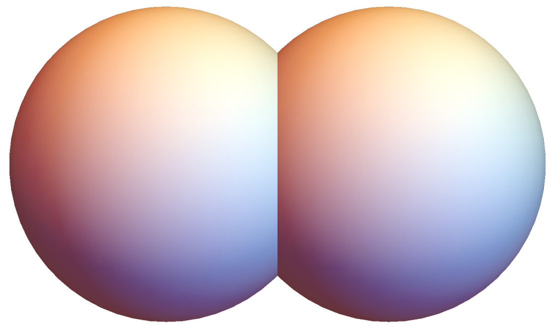
Rules: No sources other than **class notes**, **homework** and **homework solutions**, may be consulted between start of the exam and when the exam is submitted at lecture, Tuesday October 17. Electronic computational aids (laptops, smartphones, etc.) are also not allowed.

If some of the problems seem too easy, and you suspect them of being trick questions, relax: they *are* easy. The spirit of this exam is not unlike the rigorous exams given to emergency medical technicians. For certification, EMTs are required to get nearly perfect scores — we expect nothing less of our first responders. Your performance on this exam will be similar. A perfect or near perfect score will demonstrate that you are ready to respond, with confidence and mastery, to any electrodynamics emergency that may come your way.

The Cornell Code of Academic Integrity is in effect, of course, as always.

1. A conductor has the shape of two interpenetrating spheres:



Both spheres have radius a and their center-to-center separation is $\sqrt{2}a$, with the result that along the circle of intersection the two sphere-surfaces meet at $90°$. To solve the electrostatics problem, for a net charge Q on this conductor, your first instinct is of course to use Kelvin inversion.

   (a) Say your sphere of inversion also has radius a. Where would you place the center of the sphere of inversion to make the transformed electrostatics problem as easy as possible?

(b) Sketch the conducting surfaces and point charges (if any) of the transformed problem. Quantify any lengths in your sketch in terms of a.

(c) Describe the configuration of image charges whose solution would solve the transformed problem. Note: you are *not* being asked to solve the transformed problem.

2. An infinite straight "wire" has zero charge density and current density

$$\mathbf{J} = I_0 \cos(\omega t)\delta(x)\delta(y)\,\hat{\mathbf{z}},$$

where I_0 has units of current. In the Coulomb gauge, the vector potential for this source has the form

$$\mathbf{A}(x, y, z, t) = \mathrm{Re}\left(f(r)e^{-i\omega t}\right)\hat{\mathbf{z}},$$

where $r = \sqrt{x^2 + y^2}$ and the phase of the function $f(r)$, asymptotically for large r, advances with increasing r (outgoing boundary conditions).

   (a) Write expressions for the electric and magnetic fields produced by the wire in terms of the function f.

(b) Write the partial differential equation satisfied by f.

(c) Express f as an integral.

(d) For what frequencies ω, if any, does the wire radiate power? Note: you are *not* being asked to compute the power.

3. A conducting waveguide has circular cross section in the x-y plane with radius R and has infinite extent in z. The electrostatic potential Φ due to a point charge at $x = y = z = 0$ decays as

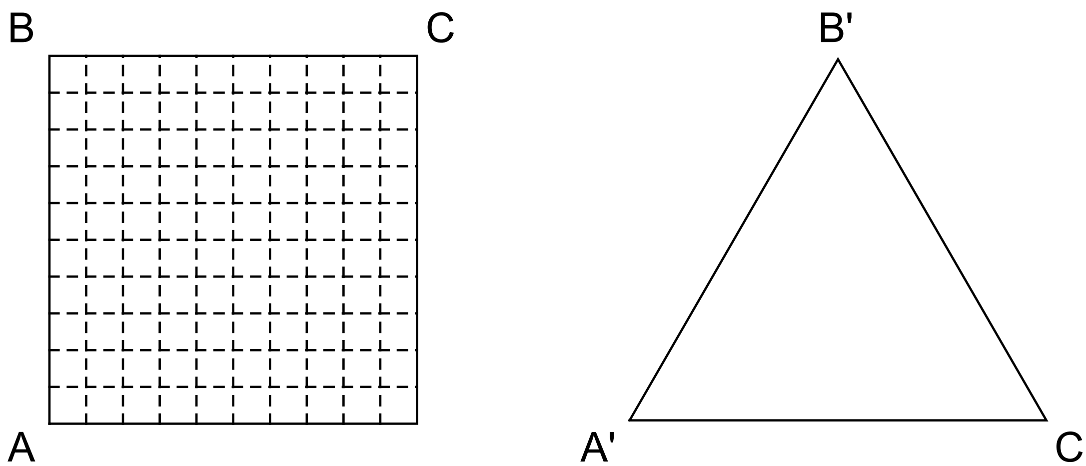
$$\Phi(x, y, z) \sim f(x, y)\, e^{-k|z|}, \qquad z \to \pm\infty.$$

   (a) The constant k is the smallest zero of the function (choose one) ...

   - $\cos(kR)$
   - $J_0(kR)$ (standard $D = 2$ Bessel function)
   - $\sin(kR)/(kR)$

   (b) Up to an overall constant, write down the function f.

4. The interior of a square is mapped conformally to the interior of an equilateral triangle so that vertices A, B, C of the square map to vertices A′, B′, C′ (respectively) of the triangle as shown:



Recall that by the Riemann mapping theorem, the mapping of three boundary points uniquely determines the map. Sketch the image of the square's grid lines on the triangle under this map. Your sketch will be evaluated on its qualitative merits rather than quantitative accuracy. You may want to make some practice sketches first!

5. The charge density in a region of space-time has the form

$$\rho(x, y, z, t) = D\,\delta'(x)\delta(y)\delta(z)\theta(t),$$

where D has units of electric-dipole moment, δ' is the derivative of the standard delta function, and $\theta(t) = 1$ for $t > 0$ and is zero otherwise (a step function). This ρ models the spontaneous appearance of a charge dipole, aligned along x, at time $t = 0$.

   (a) Calculate the current density $\mathbf{J}$, assuming it is purely along $\hat{\mathbf{x}}$, and non-zero only in the neighborhood of $x = y = z = 0$.

(b) In the Coulomb gauge, calculate the instantaneously propagated scalar potential Φ for the source ρ.

(c) Also in the Coulomb gauge, the source for the wave equation for the vector potential $\mathbf{A}$ is the transverse part of the current density, $\mathbf{J}_{\mathrm{T}}$. Calculate $\mathbf{J}_{\mathrm{T}}$.

(d) Argue, **without detailed calculations**, that the electric field for this source as calculated in the Coulomb gauge can still be causal even though the scalar potential propagates instantaneously.

6. Consider the following functional of the vector potential:

$$G[\mathbf{A}] = \int d^3x \; \mathbf{A}(\mathbf{x}) \cdot \nabla \times \mathbf{A}(\mathbf{x}).$$

(a) Is G gauge invariant? Does your answer assume any conditions on the behavior of the gauge function at infinity?

(b) Compute the following functional derivative:

$$\partial_{A_i(\mathbf{y})} \, G[\mathbf{A}] = \cdots$$