# Lecture 9: February 15

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## 9.1 Lagrangian mechanics

The single scalar function $\mathcal{L} = T - V$ contains all the information we need to produce the equations for time-evolving the generalized coordinates. This system of second-order-in-time differential equations is called the "equations of motion", and sometimes the "Euler-Lagrange" equations. The reference to Euler will emerge later in the course. Here they are, for a system with $N$ degrees of freedom:

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k}, \quad k = 1, \ldots, N. \tag{9.1}$$

Here are two things to keep in mind when working with the Lagrangian:

- The Lagrangian can only be defined when all forces are conservative, through the potential energy function $V$.
- It is important to compute $T$ and $V$ starting from coordinates in an inertial frame, since it is only in such a frame that Newton's Second Law (upon which the derivation of $\mathcal{L}$ relied) is valid.

**Remark:** There is a good reason why we do *not* write the Lagrangian equations of motion as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k}, \quad k = 1, \ldots, N, \tag{9.2}$$

even while we write Newton's Second Law as $\dot{\mathbf{p}}_i = \mathbf{F}_i$ (rather than $0 = \dot{\mathbf{p}}_i - \mathbf{F}_i$). We'll see later in the course that the two terms in the Euler-Lagrange equation are really two parts of a single entity, and setting it equal to zero is a principle of quantum mechanics.

**Example:** Let's write down the Lagrangian equations for the frictionless ladder from previous lectures. For variety, suppose this time the ladder is nearly massless and a point-like firefighter of mass $m$ is perched on the middle rung. The kinetic energy now is purely the translational kinetic energy of the firefighter:

$$T = \frac{1}{2}m\left((L/2)^2\,\dot{\theta}^2 + L\dot{\theta}\dot{w}\cos\theta + \dot{w}^2\right). \tag{9.3}$$

The only non-constraint force acting on our system is the force of gravity acting on the point mass $m$. The corresponding potential energy function is

$$V = mgy = mg(L/2)\cos\theta. \tag{9.4}$$

When the wall is stationary ($\dot{w} = 0$), the Lagrangian reduces to

$$\mathcal{L} = \frac{1}{8} m L^2 \dot{\theta}^2 - \frac{1}{2} m g L \cos\theta. \tag{9.5}$$

Writing down the equations of motion is now automatic:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{1}{4} m L^2 \ddot{\theta}, \qquad \frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} m g L \sin\theta. \tag{9.6}$$

The Euler-Lagrange equations assert that these are equal. After cancelling some common factors,

$$\ddot{\theta} = \left(\frac{g}{L/2}\right) \sin\theta. \tag{9.7}$$

**Puzzle:** This is the equation of an inverted pendulum. Can you come up with a geometrical construction (refer to the ladder diagram) that leads directly to this conclusion?

### 9.1.1 The Hamiltonian

Starting with the function $\mathcal{L}$ we can define another function $\mathcal{H}$, the *Hamiltonian*, that has the nice property of being constant in time under very general conditions:

$$\mathcal{H} = \sum_{k=1}^{N} \dot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \mathcal{L}. \tag{9.8}$$

We get three terms when computing the time derivative of $\mathcal{H}$:

$$\frac{d\mathcal{H}}{dt} = \sum_{k=1}^{N} \left( \ddot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right)\right) - \frac{d\mathcal{L}}{dt}. \tag{9.9}$$

The time derivative on the third term takes the following form when we apply the chain rule:

$$\frac{d\mathcal{L}}{dt} = \sum_{k=1}^{N} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial \mathcal{L}}{\partial q_k} \dot{q}_k \right) + \frac{\partial \mathcal{L}}{\partial t}. \tag{9.10}$$

By the Euler-Lagrange equations, the second term of (9.9) is equal to the second term of (9.10). Subtracting $d\mathcal{L}/dt$ in equation (9.9) therefore has the result that everything except one term cancels:

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{9.11}$$

From this we see that $\mathcal{H}$ is constant in time when $\mathcal{L}$ does not depend explicitly on time — when the environment is time-independent. Quantities that are constant in time are said to be “conserved”.

### 9.1.2 Conservation of $\mathcal{H}$

In one of your homework problems you will show that when there is no time-dependence in the constraint equations (the equations relating particle positions and generalized coordinates)

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_N), \tag{9.12}$$

then $\mathcal{H} = T + V$, the total mechanical energy $E$. So in this case $\mathcal{H}$ and $E$ are equal and conserved.

It's possible for the $\mathcal{H}$ constructed from $\mathcal{L}$ to be conserved and yet not equal to $E$. By the general result quoted above, for this to happen the constraint equations must have the form

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_N; t). \tag{9.13}$$

However, we do not want $t$ to appear in $\mathcal{L}$, otherwise $\partial\mathcal{L}/\partial t \neq 0$ and $\mathcal{H}$ will not be conserved. The case of a particle in an inertial moving frame satisfies both of these conditions. Let $x'$ be position (in one dimension) of the particle relative to the origin of the moving frame. This will be our single generalized coordinate. The position of the particle in the space frame is $x$, and the coordinates are related by

$$x = x' + vt, \tag{9.14}$$

where $v$ is the relative velocity of the frames (a fixed parameter). There is no potential energy, and therefore

$$\begin{aligned}
\mathcal{L} &= T && (9.15)\\
&= \frac{1}{2}m\dot{x}^2 && (9.16)\\
&= \frac{1}{2}m\left(\dot{x'}^2 + 2\dot{x'}v + v^2\right). && (9.17)\\
&= \mathcal{L}(x'; \dot{x'}). && (9.18)
\end{aligned}$$

Because this Lagrangian has no explicit $t$, the corresponding Hamiltonian will be constant in time.

**Exercise:** From definition (9.8) show that for the Lagrangian (9.17) the Hamiltonian function is

$$\mathcal{H} = \frac{1}{2}m(\dot{x'}^2 - v^2). \tag{9.19}$$

This differs from the mechanical energy $E$ by a constant, as it must because both $\mathcal{H}$ and $E$ are conserved.

Is it possible to have $\mathcal{H} = E$, but such that neither is conserved? Again, we need time-dependence in the constraint equations (9.13) except now we *want* time to appear in $\mathcal{L}$ so $\partial\mathcal{L}/\partial t \neq 0$. A simple example of this is a particle in one dimension moving in a time-dependent potential. Taking the position $x$ in the space frame as the generalized coordinate, the Lagrangian for this system is

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - U(x, t). \tag{9.20}$$

**Exercise:** Again using the definition (9.8), show that

$$\mathcal{H} = \frac{1}{2}m\dot{x}^2 + U(x,t) = E. \tag{9.21}$$

Now $\mathcal{H} = E$ but neither is conserved, such as when the particle is at rest at the minimum of $U$ while the value of $U$ at the minimum changes with time.

This is a good time to revisit a favorite freshman-physics energy conservation problem. Suppose a mass $m$, starting from rest, slides down a frictionless hill of height $h$ and then along a horizontal surface. Using conservation of energy, the final velocity $v_{\mathrm{f}} = \sqrt{2gh}$ of the mass is found by solving

$$mgh = \frac{1}{2}mv_{\mathrm{f}}^2. \tag{9.22}$$

**Puzzle:** Check energy conservation, but in the inertial frame co-moving with the mass after it is moving horizontally, with velocity $v_{\mathrm{f}}$. Why is energy not conserved?