Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 8: February 13

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8.1 The Lagrangian reformulation of mechanics

In this lecture we complete the construction begun last lecture that featured generalized coordinates, velocities and forces. The result is a new entity L called the "Lagrangian". While different mechanical systems have different L's, the mathematical procedure for obtaining a system's equations of motion from L is always the same.

8.1.1 A kinetic energy identity

The principle that we used to define generalized forces was work, a scalar quantity. We now consider another scalar quantity, the kinetic energy.

Our system is comprised of point masses m_i at positions $\mathbf{r}_i$. In the last lecture we saw how the particle velocities $\dot{\mathbf{r}}_i$ are expressed in terms of the generalized coordinates and their velocities. From this information we know that the kinetic energy always takes the following form:

$$
\begin{aligned}
T &= \sum_i \frac{1}{2} m_i\, \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i && (8.1)\\
&= T(q_1, \ldots, q_N\,; \dot{q}_1, \ldots, \dot{q}_N\,; t). && (8.2)
\end{aligned}
$$

Exercise: Express[1] T for the ladder problem of lecture 7 in terms of θ, $\dot{\theta}$, parameters L (length of ladder), m (mass of the ladder, distributed uniformly over its length), and the position of the wall, $w(t)$.

[1] Given a limited palette of mathematical symbols, duplicate usage is a fact of life we learn to come to terms with. Context is normally enough to avoid confusion, such as L for the ladder length and the symbol for the Lagrangian function.

We now derive an important identity involving partial derivatives of the kinetic energy. The first derivative is with respect to a generalized coordinate:

$$\frac{\partial T}{\partial q_k} = \sum_i m_i\,\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \tag{8.3}$$

$$= \sum_i \mathbf{p}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k}. \tag{8.4}$$

Here $\mathbf{p}_i$ is the (freshman physics) momentum of point mass i. The next derivative is with respect to a generalized velocity:

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i \mathbf{p}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \tag{8.5}$$

$$= \sum_i \mathbf{p}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}. \tag{8.6}$$

In the second line we used the identity derived in lecture 7. The identity we seek involves the total time derivative of the partial derivative we just computed:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \left(\dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \mathbf{p}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k}\right). \tag{8.7}$$

For the second term we can substitute the first partial derivative of T we obtained, equation (8.4). For the first term we invoke a principle of mechanics: Newton's Second Law:

$$\dot{\mathbf{p}}_i = \mathbf{F}_i. \tag{8.8}$$

Making both substitutions we obtain,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \frac{\partial T}{\partial q_k} \tag{8.9}$$

$$= f_k + \frac{\partial T}{\partial q_k}, \tag{8.10}$$

where the first term we recognize as the generalized force defined in lecture 7. In Newton's law (8.8) the rate of change of momentum is the net result of all forces acting on particle i, including the constraint forces. On the other hand, we observed in lecture 7 that when $\mathbf{F}_i$ appears as in (8.9), we are free to omit the constraint forces.

The identity we just derived relates the generalized forces to partial derivatives of the kinetic energy:

$$f_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}. \tag{8.11}$$

8.1.2 A potential energy identity

We now turn to the important case where the (non-constraint) forces in our system are *conservative*. This permits us to introduce another scalar, the potential energy function,

$$V = V(\mathbf{r}_1, \mathbf{r}_2, \ldots), \tag{8.12}$$

from which forces are obtained as gradients:

$$\mathbf{F}_i = -\boldsymbol{\nabla}_i V. \tag{8.13}$$

We repeat the earlier computation of partial derivatives with respect to generalized coordinates and velocities, but now for the potential. By (8.12), all dependence of V on the generalized coordinates is via the particle positions. Applying the chain rule, we obtain

$$\begin{aligned}
-\frac{\partial V}{\partial q_k} &= -\sum_i \boldsymbol{\nabla}_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} && (8.14)\\
&= \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} && (8.15)\\
&= f_k. && (8.16)
\end{aligned}$$

Also, as the $\mathbf{r}_i$'s in (8.12) do not depend on the generalized velocities,

$$\frac{\partial V}{\partial \dot{q}_k} = 0. \tag{8.17}$$

8.1.3 The Lagrangian

By substituting the partial derivative of V with respect to generalized coordinates, for the generalized force in the kinetic energy identity (8.11), we arrive at a differential equation constructed only from scalars:

$$-\frac{\partial V}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}. \tag{8.18}$$

Rewriting this as

$$0 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k}, \tag{8.19}$$

and using (8.17) to insert V into the first term,

$$0 = \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_k}\right) - \frac{\partial (T-V)}{\partial q_k}, \tag{8.20}$$

we are motivated to define a new scalar, the *Lagrangian*:

$$\begin{aligned}
L &= T - V && (8.21)\\
&= L(q_1, \ldots, q_N\,; \dot{q}_1, \ldots, \dot{q}_N\,; t). && (8.22)
\end{aligned}$$

The differential equations satisfied by the Lagrangian,

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}, \tag{8.23}$$

one for each k (degree of freedom), are of second-order in time: the total time derivative generates $\ddot{q}$'s. We have achieved the goal of generating equations for the time evolution of our mechanical system that utilize a minimal set of variables.

Exercise: Write the Lagrangian $L = T - V$ for a harmonic oscillator in one dimension and verify that (8.23) produces the correct equation of motion.