Lectures 8–10

transform wave equation into moving frame:

$$\boxed{\begin{aligned} x' &= x - ut \\ t' &= t \end{aligned}} \qquad \boxed{\begin{aligned} x &= x' + ut' \\ t &= t' \end{aligned}}$$

$$\frac{\partial}{\partial t} = \left(\frac{\partial t'}{\partial t}\right)\frac{\partial}{\partial t'} + \left(\frac{\partial x'}{\partial t}\right)\frac{\partial}{\partial x'}$$

$$= \frac{\partial}{\partial t'} - u\frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial t'}{\partial x}\right)\frac{\partial}{\partial t'} + \left(\frac{\partial x'}{\partial x}\right)\frac{\partial}{\partial x'}$$

$$= 0 + \frac{\partial}{\partial x'}$$

$$\left(\frac{\partial}{\partial t}\right)^2 = \left(\frac{\partial}{\partial t'} - u\frac{\partial}{\partial x'}\right)^2 = \left(\frac{\partial}{\partial t'}\right)^2 - 2u\frac{\partial}{\partial t'}\frac{\partial}{\partial x'} + u^2\left(\frac{\partial}{\partial x'}\right)^2$$

$$\left(\frac{\partial}{\partial x}\right)^2 = \left(\frac{\partial}{\partial x'}\right)^2$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$\Downarrow$$

$$\frac{\partial^2 \psi}{\partial t'^2} = 2u\frac{\partial^2 \psi}{\partial t'\partial x'} + (v^2 - u^2)\frac{\partial^2 \psi}{\partial x'^2}$$

<u>different</u> (not invariant)

do <u>not</u> expect invariance : medium defines "preferred" frame

But if there is no medium ...

as happens in the case of EM waves.

Fortunately, there exists transformation that works!

set $v = c$ (just to remind us of EM waves)

$$x' = \gamma_u (x - ut) \qquad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}$$
$$t' = \gamma_u (t - \tfrac{u}{c^2} x)$$

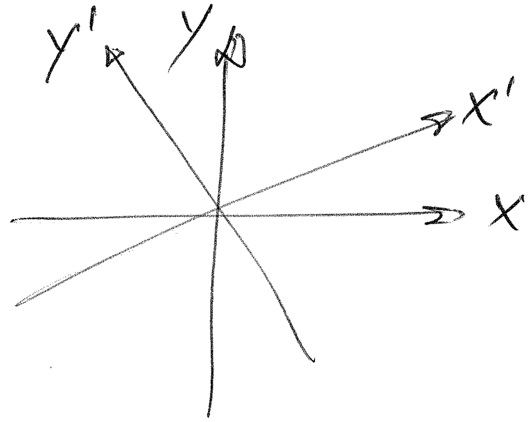
$|u| \ll c$ reduces to previous Galilean transformation

Repeat partial derivative exercise for this (Lorentz) transformation

...

Result: $$\frac{\partial^2 \psi}{\partial t'^2} = c^2 \frac{\partial^2 \psi}{\partial x'^2}$$ ✓

Invariance of $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)$ with respect to Lorentz transformations is not so different from invariance of $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \nabla^2$ with respect to <u>rotations</u>:



Laplace equation $\nabla^2 \psi = 0$ is invariant with respect to rotations.

EM waves

Gauss: $\vec{\nabla}\cdot\vec{E}=0$, $\vec{\nabla}\cdot\vec{B}=0$

Ampere: $\vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\dot{\vec{E}}$ $\quad(\mu_0\epsilon_0=1/c^2)$

Faraday: $\vec{\nabla}\times\vec{E}=-\dot{\vec{B}}$

[no sources, MKS units]

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special case:

$$E_x=0\,,\quad E_y(x,t)\,,\quad E_z=0$$

G: $\underset{x}{\underset{\uparrow}{\vec{\nabla}}}\cdot\underset{y}{\underset{\uparrow}{\vec{E}}}=0$ ✓ "transverse"

$$B_x=0\,,\quad B_y=0\,,\quad B_z(x,t)$$

G: $\underset{x}{\underset{\uparrow}{\vec{\nabla}}}\cdot\underset{z}{\underset{\uparrow}{\vec{B}}}=0$ ✓ "transverse"

$$A: \quad (\underset{\substack{\uparrow \\ x}}{\vec{\nabla}} \times \underset{\substack{\uparrow \\ z}}{\vec{B}})_y = -\frac{\partial B_z}{\partial x} = \frac{1}{c^2}\dot{E}_y$$

$$F: \quad (\underset{\substack{\uparrow \\ x}}{\vec{\nabla}} \times \underset{\substack{\uparrow \\ y}}{\vec{E}})_z = \frac{\partial E_y}{\partial x} = -\dot{B}_z$$

$$\frac{\partial}{\partial x}(F): \quad \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right)$$

$$\downarrow A$$

$$= -\frac{\partial}{\partial t}\left(-\frac{1}{c^2}\frac{\partial E_y}{\partial t}\right)$$

$$\Rightarrow c^2 \frac{\partial^2 E_y}{\partial x^2} = \frac{\partial^2 E_y}{\partial t^2}$$

1D wave equation

$\psi(x,t) \rightarrow E_y(x,t)$

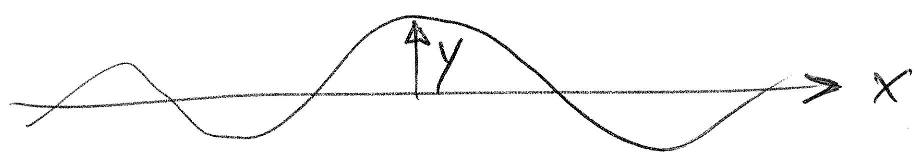
Normal Modes

time dependence : always sinusoidal

$$\psi(x,t) = \cos\omega t \; \underbrace{f_\omega(x)}_{\text{general spatial func.}}$$

Easy example (maybe too easy) :

elastic string, transverse oscillations



$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

substitute

$$y(x,t) = \cos\omega t \; f_\omega(x)$$

$$(-\omega^2 \cos\omega t)\, f_\omega(x) = v^2 \cos\omega t \,\frac{\partial^2 f_\omega}{\partial x^2}$$

$$\Rightarrow \quad -(\omega^2/v^2)\, f_\omega = \frac{\partial^2 f_\omega}{\partial x^2}$$

Solutions: $f_\omega = \sin k_\omega x\,,\ \cos k_\omega x$

where $\omega^2/v^2 = k_\omega^2$

linear equations, superposition okay

Special boundary conditions:

$$0 = y(0,t) = \cos\omega t\, f_\omega(0) \quad \text{all } t$$

("clamped" at $x=0$)

$$\Rightarrow \quad f_\omega(0) = 0 \quad \Rightarrow \quad f_\omega(x) = \sin k_\omega x$$

(or mult. by arb. constant)

finite string: boundary condition at other end, $x = L$.

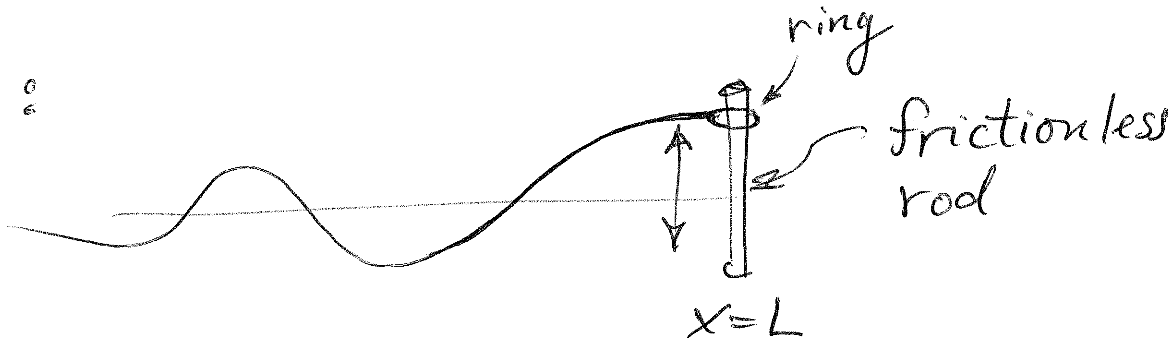
<u>clamped</u>: $0 = f_\omega(L) = \sin k_\omega L$

$$\Rightarrow \quad k_\omega L = n\pi, \quad n = 1, 2, 3, \ldots$$

restricts allowed frequencies:

$$\omega = v k_\omega = v \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

<u>free</u>:



$$0 = \left.\frac{\partial y}{\partial x}\right|_{\substack{x=L \\ \text{all } t}} = \cos\omega t \left.\left(\frac{\partial f_\omega}{\partial x}\right)\right|_{x=L}$$

$$0 = \left.\frac{\partial f_\omega}{\partial x}\right|_{x=L} = k_\omega \cos(k_\omega L)$$

$$\Rightarrow \quad k_\omega L = (n+1/2)\pi \quad , \quad n = 0, 1, 2$$

get different frequencies:

$$\omega = v k_\omega = v \frac{(n+1/2)\pi}{L} \quad , \quad n = 0, 1, 2 \ldots$$