Lecture 7

Having laid the symmetry groundwork for the electric field of an infinite line charge in lecture 6, the Gauss's law finale is easy:

\[ \vec{E} = \frac{E(r)}{r} \hat{r} \]  
(radial; scalar magnitude only dependent on \( r \))

choose \( S \) with cylindrical symmetry:

\[ \lambda L \]

\( r \)

\( S \)

\( \lambda \) charge

\( \frac{\text{length}}{} \)
\[ \vec{E} \cdot d\vec{a} = 0 \text{ on cylinder ends} \quad (2) \]

\[ \vec{E} \cdot d\vec{a} = E(r) |d\vec{a}| \text{ on curved part of surface} \]

\[ \oint \vec{E} \cdot d\vec{a} = E(r) \int |d\vec{a}| = E(r) 2\pi r L \text{ (curved surface)} \]

Gauss's law:

\[ E(r) 2\pi r L = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda \cdot L}{\varepsilon_0} \]

\[ \Rightarrow \quad E(r) = \frac{1}{2\pi \varepsilon_0} \frac{\lambda}{r} \]

The \( 1/r \) decay of \( \vec{E} \) with distance from a line charge is also the decay law if the "space" was a 2D plane and Gauss's law was
appropriately modified

Gauss's law in 2D "world":

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon'}$$

$C = \text{closed curve (1D)}$

$Q_{\text{enc}} = \text{charge enclosed by} \ C$

$d\mathbf{a} = \text{line element of} \ C$

$\varepsilon' = \text{physical constant}$
Our next goal is to express the content of Gauss's law in local terms. As a first step we need to recall a general relationship between particular kinds of integrals in D-dimensions and (D-1)-dimensions:

\[ \int_{a}^{b} \frac{df}{dx} \, dx = f(b) - f(a) \]

\(\text{D}=1\) integral of a derivative

\(\text{D}=0\) "integral" associated with boundary of \(\text{D}=1\) integral

"outward" \(\downarrow\) "inside" \(\uparrow\) "outward"
$f$ is actually a vector field, although in 1 dimension there is only 1 component so we don't need to express this with the notation. Let's see how things generalize in one higher dimension:

$$\int \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) dx dy = ?$$

$A$

This $D=2$ integral is performed inside a 2D region $A$. To keep things simple, let $A$ be a rectangle.
In our 2D integral we can trivially "do" one of the two integrals with the following result:

\[
\text{(2D integral)} = \int_{ay}^{by} \int_{bx}^{ax} f_x(x, y) \, dy \\
- \int_{by}^{ay} \int_{bx}^{ax} f_x(x, y) \, dy + \int_{ax}^{bx} \int_{ay}^{ax} f_y(x, y) \, dx \\
- \int_{ax}^{bx} \int_{ay}^{ax} f_y(x, y) \, dx = \int (f_x \hat{x} + f_y \hat{y}) \cdot d\vec{a} \\
\text{boundary (A)}
\]
This is a correct statement if we can define a vector element \( \text{d} \mathbf{a} \) on the 1D boundary of \( A \). Well, consider this:

\[
\text{d} \mathbf{a} = \hat{y} |\text{d} \mathbf{a}| \\
\text{d} \mathbf{a} = \hat{x} |\text{d} \mathbf{a}| \\
\text{d} \mathbf{a} = -\hat{y} |\text{d} \mathbf{a}| \\
\text{d} \mathbf{a} = \hat{x} |\text{d} \mathbf{a}| \\
\]

These definitions—complete with signs—correctly reproduce our 4 boundary integrals. (\(|\text{d} \mathbf{a}| = dx\) or \(dy\), whichever direction we're integrating.) But these \( \text{d} \mathbf{a} \)'s are exactly the outward normal surface/boundary elements.
Summarizing, in more compact notation,

\[ \int_A (\nabla \cdot \mathbf{f}) \, dx\,dy = \oint_{\text{boundary}(A)} \mathbf{f} \cdot d\mathbf{a} \]

where \( \nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \) is the "divergence".

This generalizes in the obvious way to any number of dimensions. We can also see why it works for more complex regions \( A \) by combining rectangular ones:
\[ \oint \vec{f} \cdot d\vec{a} + \oint \vec{f} \cdot d\vec{a} = \]

\[ \text{bound}(A_1) \quad \text{bound}(A_2) \]

\[ \oint \vec{f} \cdot d\vec{a} \quad \text{bound}(A) \]

(integrals over common boundary cancel because \( d\vec{a} \)'s have opposite sign)