Physics 3318: Analytical Mechanics	**Spring 2017**

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6.1 Rigid body dynamics (continued)

6.1.1 The symmetric top

The Euler equations are still interesting, but simplify significantly, when two of the principal moments are equal, say $I_1 = I_2 = I$. When the third moment is greater than the other two, $I_3 > I$, the symmetric top is said to be "oblate". In the other case, $I_3 < I$, the symmetric top is "prolate". The special axis usually coincides with a symmetry axis of the body. For example, a ring is an oblate symmetric top with the $\hat{\mathbf{3}}$ axis coinciding with the axis of rotational symmetry. A square plate has lower symmetry (only rotations in multiples of 90°) and yet like the ring is a symmetric top in that all the moments about axes in the plane of the plate are equal. Oblate bodies produce more angular momentum (per unit angular velocity) when rotated about $\hat{\mathbf{3}}$ than when rotated about any axis perpendicular to $\hat{\mathbf{3}}$. The opposite holds for prolate bodies. Famed football[1] coach John Heisman was fond of referring to the football as a "prolate spheroid".

Rather than specialize the Euler equations (lecture 5) to symmetric tops, we will start from scratch and keep our equations in the form of vector equations[2]. The angular momentum for symmetric tops can be written as

$$\mathbf{L} = I\boldsymbol{\omega} + (I_3 - I)\omega_3\,\hat{\mathbf{3}}. \tag{6.1}$$

This shows that the three vectors, $\mathbf{L}$, $\boldsymbol{\omega}$ and $\hat{\mathbf{3}}$ always lie in a common plane (since scalar multiples of two give the third). When taking the time derivative of this equation we make one reference to the third Euler equation, namely that $\dot{\omega}_3 = 0$ when $I_1 = I_2$. Using this fact, we obtain

$$\dot{\mathbf{L}} = 0 = I\dot{\boldsymbol{\omega}} + (I_3 - I)\omega_3\,\boldsymbol{\omega}\times\hat{\mathbf{3}}. \tag{6.2}$$

Defining the following scalar angular velocity

$$\Omega = \left(\frac{I_3 - I}{I}\right)\omega_3, \tag{6.3}$$

we can rewrite (6.2) as

$$\dot{\boldsymbol{\omega}} = (\Omega\,\hat{\mathbf{3}})\times\boldsymbol{\omega}. \tag{6.4}$$

Whereas $\Omega\,\hat{\mathbf{3}}$ is not a constant vector (it precesses around $\boldsymbol{\omega}$), it *is* a constant vector for an observer based on the body. For those observers, at least, the content of (6.4) is quite simple: the angular velocity vector precesses about $\hat{\mathbf{3}}$ at rate Ω, and in a direction that is opposite for oblate and prolate bodies (the sign of Ω).

[1] All references to "football" are to the North American sport. The "footballs" kicked around by soccer teams are spherical tops and therefore of limited interest to the serious student of rigid body motion.

[2] The Euler equations involve only scalar variables.

6.1.2 The Chandler wobble

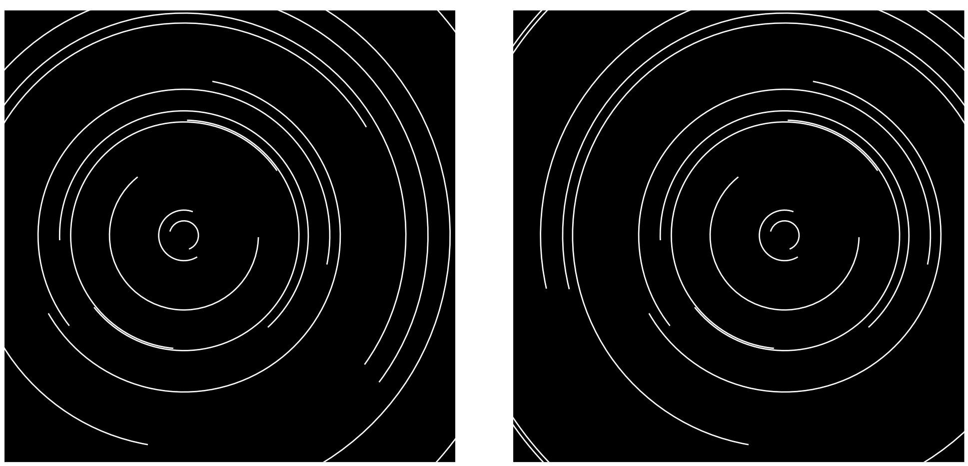
A direct application of equation (6.4) is *the wobble of the Earth, as seen from Earth.* To a rough approximation the Earth is an oblate rigid body, with

$$\frac{I_3 - I}{I} \approx \frac{1}{300}. \tag{6.5}$$

The slight excess of I_3 over I is consistent with the small increase in the equatorial radius relative to the polar radius in an equilibrium rotating viscous-fluid planet. Assuming we can neglect torques (mostly due to the moon), damping (from fluid motion in the Earth), and effects due to the core (slight excess rotation rate), the Earth's angular velocity vector $\boldsymbol{\omega}$ is predicted by equation (6.4) to maintain a fixed angle relative to the Earth's symmetry axis ($\hat{\mathbf{3}}$) as it precesses about this axis. The precession period,

$$T = \frac{2\pi}{\Omega} \approx \frac{2\pi}{\omega_3} \times 300 = 24\,\text{hours} \times 300, \tag{6.6}$$

is about one year. Over shorter periods, say hours, the angular velocity vector appears static and can be located very precisely by pointing a telescope at the sky. Aiming the telescope parallel to the rotation axis, a time-lapse image of the stars will show that they appear to orbit about a point in the sky — the Earth's "instantaneous" rotation axis. This point will have shifted when another image is made a few months later, with the same telescope pointed exactly as before:



The maximum shift (after a half period) is comparable to the angular diameter of Neptune, about one micro-radian. It was first detected by the American astronomer Seth Carlo Chandler in 1891. The observed period, about 430 days, is a bit longer than our estimate, indicating that the Earth deviates from the strict rigid-body model on the time scale of the measurement.

6.1.3 The symmetric top in the space frame

The description of rigid body motion in the space frame is easiest for the symmetric top. Imagine a football tumbling through space: a prolate rigid body with $\hat{\mathbf{3}}$ along the long diameter of the ball. Since $\hat{\mathbf{3}}$ is fixed in the body, its time derivative is given by

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega} \times \hat{\mathbf{3}}. \tag{6.7}$$

However, this is not all that helpful since the precession axis $\boldsymbol{\omega}$ is itself not static! Fortunately, by the symmetric top identity (6.1) we can express the $\boldsymbol{\omega}$ in (6.7) in terms of two other vectors:

$$\dot{\hat{\mathbf{3}}} = \left(\mathbf{L}/I - \left(\frac{I_3 - I}{I}\right)\omega_3\,\hat{\mathbf{3}}\right) \times \hat{\mathbf{3}} = \frac{1}{I}\mathbf{L} \times \hat{\mathbf{3}}. \tag{6.8}$$

This is better because for torque-free motion $\mathbf{L}$ is a static vector. Defining the constant (vector) angular velocity,

$$\boldsymbol{\omega}_p = \frac{1}{I}\mathbf{L}, \tag{6.9}$$

the motion of $\hat{\mathbf{3}}$ is simple precession with respect to it:

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega}_p \times \hat{\mathbf{3}}. \tag{6.10}$$

We now derive a simple formula for the magnitude of the precession rate, $\omega_p = L/I$. On the one hand we know that

$$\mathbf{L} \cdot \hat{\mathbf{3}} = I_3\,\omega_3 \tag{6.11}$$

is a constant because ω_3 is a constant by Euler's third equation (since $I_1 = I_2$). On the other hand, we can express this constant in terms of the fixed angle — the wobble angle — between $\hat{\mathbf{3}}$ and the precession axis (the axis of $\mathbf{L}$):

$$\mathbf{L} \cdot \hat{\mathbf{3}} = L\cos\theta. \tag{6.12}$$

Equating the two expressions for $\mathbf{L} \cdot \hat{\mathbf{3}}$ we obtain

$$\omega_p = L/I = \frac{I_3\,\omega_3}{I\cos\theta}. \tag{6.13}$$

Suppose a football is thrown with a $\theta = 25°$ wobble, so $\cos\theta \approx 0.9$. Using this and $I_3/I = 0.6$ (regulation football) in equation (6.13), we then find

$$\omega_3/\omega_p = (I/I_3)\cos\theta = 3/2. \tag{6.14}$$

You might conclude from this that the axis precesses twice in the time the football spins three times about its axis. But what exactly does it mean for the football to spin about its axis, when the axis itself is in motion (precessing)?

A more precise question is the following: by what angle ϕ_1 has the body axis $\hat{\mathbf{1}}$ of the football rotated about $\hat{\mathbf{3}}$ (the symmetry axis) after one precession period (so $\hat{\mathbf{3}}$ is back where it started)? We will calculate ϕ_1 using what we have learned about free precession, both when viewed from the body frame and when viewed from the inertial space frame. To follow the calculation you should refer to the diagram on the next page.

The vectors $\mathbf{L}$, $\boldsymbol{\omega}$ and $\hat{\mathbf{3}}$ always lie in a plane; in the diagram we see them at time $t = 0$. After one precession period, $T_p = 2\pi/\omega_p$, we get exactly the same set of three vectors because $\mathbf{L}$ is fixed, $\boldsymbol{\omega}$ has returned to its original position after one precession period, and $\hat{\mathbf{3}}$ is uniquely determined by the other two. We define the body axis $\hat{\mathbf{1}}$ of the football as the projection of $\boldsymbol{\omega}$, at $t = 0$, onto the plane perpendicular to $\hat{\mathbf{3}}$. Because the vector $\hat{\mathbf{1}}$ might not return to it original position after one period, the diagram shows $\hat{\mathbf{1}}(0)$ and $\hat{\mathbf{1}}(T_p)$.

Let's start by working in the body frame. In this frame the football — in particular $\hat{\mathbf{3}}$ and $\hat{\mathbf{1}}(0)$ — are fixed and $\boldsymbol{\omega}$ precesses at rate Ω about the $\hat{\mathbf{3}}$. Because the football is prolate, Ω is negative and the direction is clockwise. Therefore, in the body frame, $\boldsymbol{\omega}$ rotates (clockwise) about $\hat{\mathbf{3}}$ with angle

$$|\Omega|T_p = 2\pi\frac{|\Omega|}{\omega_p} = 2\pi\left(\frac{I - I_3}{I_3}\right)\cos\theta. \tag{6.15}$$

Using the same parameter values as above, the clockwise rotation angle is 218°. The position of $\boldsymbol{\omega}$ after this rotation is marked $\boldsymbol{\omega}'$ in the diagram.

But in the space frame we know that $\boldsymbol{\omega}$ returns exactly to where it started after one precession period T_p! This is only possible, in our diagram, if the body axes $\hat{\mathbf{1}}$ and $\hat{\mathbf{2}}$ do not return to their starting positions but end up rotated about the $\hat{\mathbf{3}}$ axis. The diagram shows the rotated basis vector $\hat{\mathbf{1}}(T_p)$; it is obtained by rotating the body frame by clockwise angle

$$\phi_1 = 2\pi - |\Omega| T_p, \tag{6.16}$$

or 142°. When this rotation of the body axes is combined with the rotation by $|\Omega|T_p$ of $\boldsymbol{\omega}$ relative to the body, the result is that $\boldsymbol{\omega}$ makes a complete orbit.

What's interesting about ϕ_1 is that — aside from special values of I_3/I and $\cos\theta$ — it is not a rational multiple of 2π. When ϕ_1 is an irrational multiple of 2π, then no matter how many complete precessions of the $\hat{\mathbf{3}}$ axis we allow, the $\hat{\mathbf{1}}$ axis (position of the stitches) will never exactly return to where it started out, though it will come arbitrarily close. Motion with these characteristics is called "quasiperiodic".