5.1 Rigid body dynamics (continued)

While one of our goals is to describe the complex motions of rigid bodies in space, it turns out that a simpler problem is to describe this motion as seen in the frame of the tumbling/rotating body itself! The equations of motion in this frame are called the Euler rigid body equations. After we derive these equations and apply them to the slight wobble of the Earth’s motion, we return to the actual problem of rigid motion as seen in the space frame.

5.1.1 Rigid body motion from the body frame perspective

We start by writing the equation of the body’s angular momentum

$$ L = I \cdot \omega, \quad (5.1) $$

in terms of the special body-fixed basis, the principal basis. In this basis

$$ I = I_1 \hat{1} \hat{1} + I_2 \hat{2} \hat{2} + I_3 \hat{3} \hat{3}, \quad (5.2) $$

and

$$ \omega = \omega_1 \hat{1} + \omega_2 \hat{2} + \omega_3 \hat{3}. \quad (5.3) $$

While the principal moments $I_1$, $I_2$ and $I_3$ are time-independent, the components of the angular velocity do not have this property. Substituting these expressions into (5.1), we obtain

$$ L = I_1 \omega_1 \hat{1} + I_2 \omega_2 \hat{2} + I_3 \omega_3 \hat{3}. \quad (5.4) $$

The dynamical basis of the equations of motion is the constancy of $L$ in the absence of torques. We should therefore take the time derivative of the right hand side of (5.4) and set it equal to zero. In addition to time derivatives of the components, $\dot{\omega}_1$, we also need to account for the time derivatives of the basis vectors:

$$ \dot{\hat{1}} = \omega \times \hat{1}, \quad (5.5) $$

$$ \dot{\hat{1}} = (\omega_1 \hat{1} + \omega_2 \hat{2} + \omega_3 \hat{3}) \times \hat{1} \quad (5.6) $$

$$ = \omega_3 \hat{2} - \omega_2 \hat{3}. \quad (5.7) $$

Including both forms of time derivative we obtain

$$ \dot{L} = 0 = I_1 \dot{\omega}_1 \hat{1} + I_2 \dot{\omega}_2 \hat{2} + I_3 \dot{\omega}_3 \hat{3} + I_1 \omega_1 (\omega_3 \hat{2} - \omega_2 \hat{3}) + I_2 \omega_2 (\omega_1 \hat{3} - \omega_3 \hat{1}) + I_3 \omega_3 (\omega_2 \hat{1} - \omega_1 \hat{2}). \quad (5.8) $$
This is really a set of three equations, since the net coefficient of each basis vector must vanish:

\[
\begin{align*}
I_1 \dot{\omega}_1 &= (I_2 - I_3)\omega_2 \omega_3 \quad (5.9) \\
I_2 \dot{\omega}_2 &= (I_3 - I_1)\omega_3 \omega_1 \quad (5.10) \\
I_3 \dot{\omega}_3 &= (I_1 - I_2)\omega_1 \omega_2. \quad (5.11)
\end{align*}
\]

This system of first-order, non-linear differential equations are called *Euler’s rigid body equations*.

We now have a mathematical procedure for determining the time evolution of the rigid body. Given any angular momentum vector \( \mathbf{L} \) of the body, and an arbitrary initial orientation as specified by 1, 2 and 3, the initial components \( \omega_1(0) \), \( \omega_2(0) \) and \( \omega_3(0) \) are determined by (5.4). These are the initial conditions for the Euler equations which give us \( \omega_1(t) \), \( \omega_2(t) \) and \( \omega_3(t) \) at future times.

### 5.1.2 Properties of the solutions to Euler’s equations

If you take the three Euler equations and add them together after multiplying them, respectively, by \( \omega_1 \), \( \omega_2 \) and \( \omega_3 \), the right hand side is zero. This fact is a reminder to us that however complicated the time evolution of \( \omega \) may be, the three components of the angular velocity will always lie on the surface of the energy ellipsoid

\[
T_{\text{rot}} = \frac{1}{2} \mathbf{\omega} \cdot \mathbf{I} \cdot \mathbf{\omega} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2.
\]

The Euler equations for a *spherical top* are especially simple. A spherical top is a body with all principal moments equal. From Euler’s equations we see that all three components of the angular velocity have vanishing time derivatives when \( I_1 = I_2 = I_3 \), or in symbols: \( \dot{\omega} = 0 \). And since \( \dot{\omega} = \omega \) is always true, \( \dot{\omega} = 0 \) implies that the angular velocity vector of a spherical top is constant.

Now consider a general rigid body but the very special initial condition, where two of the principal components of the angular velocity are exactly zero, say \( \omega_1(0) = \omega_2(0) = 0 \). Again we see (from Euler’s equations) that all the right hand sides are zero, so that the evolution is trivial (two components stay zero and the third is constant, not necessarily zero). However, we should always be cautious in physics when a certain conclusion rests on something being “exactly zero”. The better approach in such situations is to use perturbation analysis.

Suppose that \( \omega_1 \) and \( \omega_2 \) are both very small, though not necessarily zero. The third Euler equation then tells us that the time derivative of \( \omega_3 \) is then even smaller, that is, of second order in the magnitudes of \( \omega_1 \) and \( \omega_2 \). At our first order level of approximation we may then approximate \( \omega_3 \) as a constant. The first two Euler equations in that approximation reduce to linear equations. Taking the time derivative of the first,

\[
I_1 \ddot{\omega}_1 \approx (I_2 - I_3)\omega_2 \omega_3,
\]

and substituting the time derivative of \( \omega_2 \) from the second equation,

\[
I_2 I_1 \ddot{\omega}_1 \approx (I_2 - I_3)(I_3 - I_1)\omega_3^2 \omega_1,
\]

we obtain a linear differential equation involving only \( \omega_1 \). For this equation to have stable (bounded) solutions we need the factor \((I_2 - I_3)(I_3 - I_1)\) to be negative, and this will be the case when \( I_3 \) is either the smallest or the largest principal moment. If instead \( I_3 \) has a value between the other two, so \((I_2 - I_3)(I_3 - I_1)\) is
positive, then the solution will diverge exponentially no matter how small the initial conditions. Moreover, when \((I_2 - I_3)(I_3 - I_1)\) is negative and \(\omega_1\) is a bounded sinusoidal function, then the same holds true for \(\omega_2\) by the first Euler equation, which relates \(\omega_2\) to the time derivative of \(\omega_1\). Expanding this analysis to the other three cases of special (and perturbed) initial conditions, we see that only two of the three cases will be stable, where the unstable case corresponds to the axis whose moment is the one in the middle.

All the remarks above are on display in the following diagram showing the “orbits” of the \(\omega\) evolution on the energy ellipsoid:

Stable circular orbits surround the shortest and longest axes of the ellipsoid, corresponding to, respectively, the smallest and largest principal moments.

**Drawing exercise:** Add axes labeled \(\hat{1}, \hat{2}\) and \(\hat{3}\) for the case \(I_1 < I_2 < I_3\). Add arrows to the orbits showing the direction of the time evolution (use the Euler equations near the axes as a guide).

### 5.1.3 Systematic perturbation analysis of the Euler equations

To support the claim that the non-constant part of \(\omega_3\) is small to second order, when \(\omega_1\) and \(\omega_2\) are small to first order, we write

\[
\begin{align*}
\omega_1(t) &= \epsilon a(t) + \cdots & (5.16) \\
\omega_2(t) &= \epsilon b(t) + \cdots & (5.17) \\
\omega_3(t) &= c + \epsilon^2 d(t) + \cdots , & (5.18)
\end{align*}
\]

where \(\epsilon\) is a dimensionless small parameter and \(\cdots\) denotes terms having higher powers of \(\epsilon\). To check the consistency of this perturbation expansion, we substitute these expressions into the Euler equations keeping
only lowest order terms:

\[
\begin{align*}
\epsilon I_1 \dot{a}(t) &= \epsilon (I_2 - I_3)b(t)c \\
\epsilon I_2 \dot{b}(t) &= \epsilon (I_3 - I_1)c a(t) \\
\epsilon^2 I_3 \dot{d}(t) &= \epsilon^2 (I_1 - I_2)a(t)b(t).
\end{align*}
\] (5.19, 5.20, 5.21)

The first two equations for \( a(t) \) and \( b(t) \) are coupled and linear, and have sinusoidal (bounded) solutions when \((I_2 - I_3)(I_3 - I_1)\) is negative, as discussed earlier. From the third equation we see that \( d(t) \) is obtained by integrating the product of the solutions \( a(t) \) and \( b(t) \). Finally, note that because \( a(t) \) and \( b(t) \) are always 90° degrees out of phase, their product is equally often positive as it is negative. Therefore, the integral of their product, \( d(t) \), will also be sinusoidal and stay bounded.