Lecture 5: February 6

Instructor: Veit Elser

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

5.1 Rigid body dynamics (continued)

While one of our goals is to describe the complex motions of rigid bodies in space, it turns out that a simpler problem is to describe this motion as seen in the frame of the tumbling/rotating body itself! The equations of motion in this frame are called the Euler rigid body equations. After we derive these equations and apply them to the slight wobble of the Earth's motion, we return to the actual problem of rigid motion as seen in the space frame.

5.1.1 Rigid body motion from the body frame perspective

We start by writing the equation of the body's angular momentum

$$\mathbf{L} = \mathbf{I}\cdot\boldsymbol{\omega}, \tag{5.1}$$

in terms of the special body-fixed basis, the principal basis. In this basis

$$\mathbf{I} = I_1\hat{\mathbf{1}}\hat{\mathbf{1}} + I_2\hat{\mathbf{2}}\hat{\mathbf{2}} + I_3\hat{\mathbf{3}}\hat{\mathbf{3}}, \tag{5.2}$$

and

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{1}} + \omega_2\hat{\mathbf{2}} + \omega_3\hat{\mathbf{3}}. \tag{5.3}$$

While the principal moments I_1, I_2 and I_3 are time-independent by construction, the components ω_1, ω_2 and ω_3 of the angular velocity do not have this property. Substituting these expressions into (5.1), we obtain

$$\mathbf{L} = I_1\omega_1\hat{\mathbf{1}} + I_2\omega_2\hat{\mathbf{2}} + I_3\omega_3\hat{\mathbf{3}}. \tag{5.4}$$

The dynamical basis of the equations of motion is the constancy of $\mathbf{L}$ in the absence of torques. We should therefore take the time derivative of the right hand side of (5.4) and set it equal to zero. In addition to time derivatives of the components, $\dot{\omega}_1$, we also need to account for the time derivatives of the basis vectors:

$$\begin{aligned}
\dot{\hat{\mathbf{1}}} &= \boldsymbol{\omega}\times\hat{\mathbf{1}} && (5.5)\\
&= (\omega_1\hat{\mathbf{1}} + \omega_2\hat{\mathbf{2}} + \omega_3\hat{\mathbf{3}})\times\hat{\mathbf{1}} && (5.6)\\
&= \omega_3\hat{\mathbf{2}} - \omega_2\hat{\mathbf{3}}. && (5.7)
\end{aligned}$$

Including both forms of time derivative we obtain

$$\dot{\mathbf{L}} = 0 = I_1\dot{\omega}_1\hat{\mathbf{1}} + I_2\dot{\omega}_2\hat{\mathbf{2}} + I_3\dot{\omega}_3\hat{\mathbf{3}} + I_1\omega_1(\omega_3\hat{\mathbf{2}} - \omega_2\hat{\mathbf{3}}) + I_2\omega_2(\omega_1\hat{\mathbf{3}} - \omega_3\hat{\mathbf{1}}) + I_3\omega_3(\omega_2\hat{\mathbf{1}} - \omega_1\hat{\mathbf{2}}). \tag{5.8}$$

This is really a set of three equations, since the net coefficient of each basis vector must vanish:

$$I_1\,\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \tag{5.9}$$

$$I_2\,\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 \tag{5.10}$$

$$I_3\,\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2. \tag{5.11}$$

This system of first-order, non-linear differential equations are called *Euler's rigid body equations.*

We now have a mathematical procedure for determining the time evolution of the rigid body. Given any angular momentum vector $\mathbf{L}$ of the body, and an arbitrary initial orientation as specified by $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$, the initial components $\omega_1(0)$, $\omega_2(0)$ and $\omega_3(0)$ are determined by (5.4). These are the initial conditions for the Euler equations which give us $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$ at future times.

5.1.2 Properties of the solutions to Euler's equations

If you take the three Euler equations and add them together after multiplying them, respectively, by ω_1, ω_2 and ω_3, the right hand side is zero. This is not a surprise because the sum of the left hand sides

$$\omega_1 I_1\dot{\omega}_1 + \omega_2 I_2\dot{\omega}_2 + \omega_3 I_3\dot{\omega}_3 = 0, \tag{5.12}$$

is the time derivative of the rotational kinetic energy $T_{\rm rot}$, a constant. This fact is a reminder to us that however complicated the time evolution of $\boldsymbol{\omega}$ may be, the three components of the angular velocity will always lie on the surface of the energy ellipsoid

$$T_{\rm rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2. \tag{5.13}$$

The Euler equations for a *spherical top* are especially simple. A spherical top is a body with all principal moments equal. From Euler's equations we see that all three components of the angular velocity have vanishing time derivatives when $I_1 = I_2 = I_3$, or in symbols: $\mathring{\boldsymbol{\omega}} = 0$. And since $\mathring{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}$ is always true, $\mathring{\boldsymbol{\omega}} = 0$ implies that the angular velocity vector of a spherical top is constant.

Now consider a general rigid body but the very special initial condition, where two of the principal components of the angular velocity are exactly zero, say $\omega_1(0) = \omega_2(0) = 0$. Again we see (from Euler's equations) that all the right hand sides are zero, so that the evolution is trivial (two components stay zero and the third is constant, not necessarily zero). However, we should always be cautious in physics when a certain conclusion rests on something being "exactly zero". The better approach in such situations is to use perturbation analysis.

Suppose that ω_1 and ω_2 are both very small, though not necessarily zero. The third Euler equation then tells us that the time derivative of ω_3 is then even smaller, that is, of second order in the magnitudes of ω_1 and ω_2. At our first order level of approximation we may then approximate ω_3 as a constant. The first two Euler equations in that approximation reduce to linear equations. Taking the time derivative of the first,

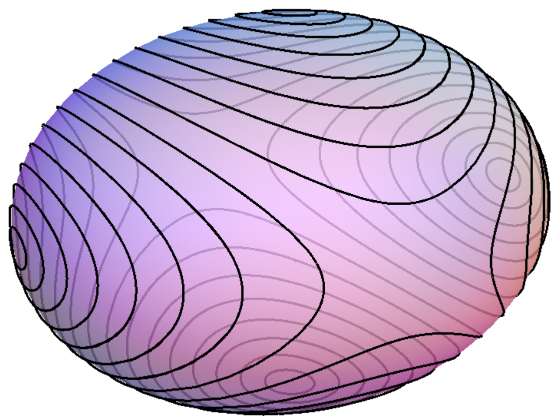
$$I_1\ddot{\omega}_1 \approx (I_2 - I_3)\dot{\omega}_2\omega_3, \tag{5.14}$$

and substituting the time derivative of ω_2 from the second equation,

$$I_2 I_1\ddot{\omega}_1 \approx (I_2 - I_3)(I_3 - I_1)\omega_3^2\,\omega_1, \tag{5.15}$$

we obtain a linear differential equation involving only ω_1. For this equation to have stable (bounded) solutions we need the factor $(I_2 - I_3)(I_3 - I_1)$ to be negative, and this will be the case when I_3 is either the smallest or the largest principal moment. If instead I_3 has a value between the other two, so $(I_2 - I_3)(I_3 - I_1)$ is positive, then the solution will diverge exponentially no matter how small the initial conditions. In the stable case, when $(I_2 - I_3)(I_3 - I_1)$ is negative and ω_1 is a bounded sinusoidal function, then the same holds true for ω_2 by the first Euler equation, which relates ω_2 to the time derivative of ω_1. Expanding this analysis to the other three cases of special (and perturbed) initial conditions, we see that only two of the three cases will be stable, where the unstable case corresponds to the axis whose moment is the one in the middle.

All the remarks above are on display in the following diagram showing the "orbits" of the $\boldsymbol{\omega}$ evolution on the energy ellipsoid:



Stable circular orbits surround the shortest and longest axes of the ellipsoid, corresponding to, respectively, the largest and smallest principal moments.

Drawing exercise: Add axes labeled $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ for the case $I_1 < I_2 < I_3$. Add arrows to the orbits showing the direction of the time evolution (use the Euler equations near the axes as a guide).

5.1.3 Systematic perturbation analysis of the Euler equations

To support the claim that the non-constant part of ω_3 is small to second order, when ω_1 and ω_2 are small to first order, we write

$$\omega_1(t) = \epsilon\, a(t) + \cdots \quad (5.16)$$
$$\omega_2(t) = \epsilon\, b(t) + \cdots \quad (5.17)$$
$$\omega_3(t) = c + \epsilon^2\, d(t) + \cdots\,, \quad (5.18)$$

where ϵ is a dimensionless small parameter and $\cdots$ denotes terms having higher powers of ϵ. To check the consistency of this perturbation expansion, we substitute these expressions into the Euler equations keeping only lowest order terms:

$$\begin{aligned} \epsilon\, I_1\, \dot{a}(t) &= \epsilon\, (I_2 - I_3) b(t) c & (5.19)\\ \epsilon\, I_2\, \dot{b}(t) &= \epsilon\, (I_3 - I_1) c\, a(t) & (5.20)\\ \epsilon^2\, I_3\, \dot{d}(t) &= \epsilon^2\, (I_1 - I_2) a(t) b(t). & (5.21) \end{aligned}$$

The first two equations for $a(t)$ and $b(t)$ are coupled and linear, and have sinusoidal (bounded) solutions when $(I_2 - I_3)(I_3 - I_1)$ is negative, as discussed earlier. From the third equation we see that $d(t)$ is obtained by integrating the product of the solutions $a(t)$ and $b(t)$. Finally, note that because $a(t)$ and $b(t)$ are always 90° degrees out of phase, their product is equally often positive as it is negative. Therefore, the integral of their product, $d(t)$, will also be sinusoidal and stay bounded.