A circuit that combines the two forms of energy storage — electric and magnetic — is a capacitor connected to an inductor:

\[
\begin{align*}
I(t) &\quad C \\
Q(t) &\quad L \\
a &\quad b
\end{align*}
\]

Initially we'll have the capacitor charged, \( Q(0) = Q_0 \), and current first starts flowing at \( t=0 \) when we close a switch, so \( I(0) = 0 \).
Applying the loop rule to our circuit:

\[ V_a - V_b = L \frac{dI}{dt} = \frac{Q}{C} \]

And, since \( I = -\frac{dQ}{dt} \),

\[ L \left( -\frac{d^2Q}{dt^2} \right) = \frac{Q}{C} \]

or \( \frac{d^2Q}{dt^2} = -\frac{Q}{LC} \)

The combination \( LC \) is, dimensionally, the square of a time:

\[ LC = t^2 \]

We also know that \( L \) and \( C \)
are $\mu_0$ and $\varepsilon_0$ times lengths associated with the device geometry:

$$L = \mu_0 \lambda_L \quad C = \varepsilon_0 \lambda_C.$$ 

$$\Rightarrow (\mu_0 \lambda_L \sqrt{\varepsilon_0 \lambda_C}) = \frac{c^2}{2}$$

$$\frac{\sqrt{\lambda_L \lambda_C}}{2} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$$

So the geometric mean of the lengths, divided by the time $T$, is the speed of light.

We can also let $1/LC$ define the square of a frequency, or angular frequency.
\[
\frac{1}{LC} = \omega^2
\]

Our equation for \( Q(t) \) is now
\[
\dot{Q} = -\omega^2 Q
\]

with general solution
\[
Q(t) = A \cos(\omega t - \phi)
\]

where \( A \) and \( \phi \) are arbitrary constants. Since we want \( \dot{Q}(0) = -I(0) = 0 \), \( \phi = 0 \), and therefore \( Q(0) = A = Q_0 \). The solution with our initial conditions is therefore
\[ Q(t) = Q_0 \cos(\omega t) \].

At times \( t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \text{ etc.} \) the charge amplitude is at a maximum or minimum and \( I = 0 \); the energy is then purely electric and stored in the capacitor:

Between these times \( Q = 0 \) and \( I \) has a maximum or minimum amplitude (clockwise vs. counter-
clockwise); the energy is then purely magnetic and stored in the inductor:

![Diagram](image)

In rough terms, the time between these extremes is $\tau$, and the distance between the devices is of order $\lambda_c$ or $\lambda_e$ or something of that order. The speed of propagation is therefore of order

$$\frac{\lambda_c}{\tau} = c.$$
We will finish up the course with Maxwell's discovery that these oscillations can happen in empty space, with energy being transformed between its electric and magnetic forms in an intricate pattern. As this is happening, energy is also flowing through space. We will finish this lecture by deriving energy flow vector field.

Consider a volume \( V \) in space with closed surface \( S \). We will compute the time-rate-of-change of electromagnetic energy inside \( V \) and relate it to a quantity
that flows through \( S \).

\[
\mathcal{U}_V = \oint_{\partial V} \left( \frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B} \right) \, d^3r
\]

\( \text{electric energy density} \quad \text{magnetic energy density} \)

\[
\frac{d\mathcal{U}_V}{dt} = \mathcal{I} = \int_V \left( \varepsilon_0 \vec{E} \cdot \frac{\partial}{\partial t} \vec{E} + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial}{\partial t} \vec{B} \right) \, d^3r
\]

Substitute \( \vec{E} \) and \( \vec{B} \) from Maxwell's equations:

\[
\frac{d\mathcal{U}_V}{dt} = \int_V \left( \varepsilon_0 \vec{E} \cdot \nabla \times \vec{B} + \frac{1}{\mu_0} \vec{B} \cdot \left( -\nabla \times \vec{E} \right) \right) \, d^3r
\]
\[
\frac{1}{\mu_0} \oint \left( \vec{E} \cdot \nabla \times \vec{B} - \vec{B} \cdot \nabla \times \vec{E} \right) d^3r
\]

This is the volume integral of a divergence, to which we can apply the divergence theorem. Define

\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B},
\]

then

\[
\frac{d\Phi}{dt} = -\oint_{\Sigma} \vec{S} \cdot d\vec{a}.
\]

Thus \( \vec{S} \) represents the flux of electromagnetic energy. Its
units are energy per unit area and time, also called "intensity". The direction of $\vec{S}$ is the direction of energy flow: when $\vec{S}$ is parallel to the outward normals $d\vec{a}$ then the surface integral is positive and energy leaves the volume. The vector field $\vec{S}$ is named after John Poynting ( $\vec{S}$ "poynts" in the direction of energy flow).