Physics 3318: Analytical Mechanics	**Spring 2017**

Lecture 4: February 3

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4.1 Rigid body dynamics

In the previous lectures we did not concern ourselves with the cause behind a body having an angular velocity $\boldsymbol{\omega}$, or why it is or is not constant in time. In those lectures we treated $\boldsymbol{\omega}$ as imposed by some external agent and worked out the kinematic consequences. Today we move beyond that limited perspective with the goal of finding laws that explain the dynamics of $\boldsymbol{\omega}$. We know from freshman physics that $\boldsymbol{\omega}$ can change with time, both in magnitude and in direction, when there are torques. What you might find surprising is that $\boldsymbol{\omega}$ changes with time even when the rigid body is completely free of forces and torques! Those symmetrical bodies and initial conditions you studied in freshman physics turn out to be the exception and are not representative of what happens in general.

4.1.1 Kinetic energy of a rigid body

Our rigid body has center of mass $\mathbf{R}$ and is comprised of masses m_i at points $\mathbf{r}_i$ relative to $\mathbf{R}$. These masses are "fixed in the body" so their velocity relative to $\mathbf{R}$ is $\boldsymbol{\omega} \times \mathbf{r}_i$. Taking into account the translational motion of the center of mass itself, the net velocity of m_i is

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{r}_i. \tag{4.1}$$

From this we get the following expression for the total kinetic energy of the rigid body,

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) && (4.2) \\ &= \frac{1}{2}M(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}) + \frac{1}{2}\sum_i m_i\,(\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i), && (4.3) \end{aligned}$$

where

$$M = \sum_i m_i \tag{4.4}$$

is the total mass of the body and we have used the center of mass property

$$0 = \sum_i m_i \mathbf{r}_i \tag{4.5}$$

to eliminate the cross term. The two terms of T are the translational and rotational kinetic energies of the body. Not surprisingly, the two parts are described by independent variables and therefore have independent dynamics. The rest of this lecture will focus on just the rotational part.

Using vector identities we can rewrite the double cross product without any cross products at all:

$$\begin{aligned}(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot\boldsymbol{\omega}\times\mathbf{r}_i &= \boldsymbol{\omega}\cdot\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i) &(4.6)\\ &= \boldsymbol{\omega}\cdot((\mathbf{r}_i\cdot\mathbf{r}_i)\boldsymbol{\omega}-(\mathbf{r}_i\cdot\boldsymbol{\omega})\mathbf{r}_i) &(4.7)\\ &= (\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r}_i\cdot\mathbf{r}_i)-(\boldsymbol{\omega}\cdot\mathbf{r}_i)(\mathbf{r}_i\cdot\boldsymbol{\omega}). &(4.8)\end{aligned}$$

The rotational kinetic energy of the rigid body now looks like this:

$$T_{\text{rot}} = \frac{1}{2}\sum_i m_i\left((\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r}_i\cdot\mathbf{r}_i)-(\boldsymbol{\omega}\cdot\mathbf{r}_i)(\mathbf{r}_i\cdot\boldsymbol{\omega})\right). \tag{4.9}$$

4.1.2 Moment of inertia tensor

The expression for T_{rot}, a scalar, contains the angular velocity, a vector, twice. The mathematical entity that combines with two vectors to form a scalar is called a tensor, or more precisely, a tensor of order 2. One of the most basic tensors is the identity tensor

$$\mathbb{I} = \hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}. \tag{4.10}$$

To understand the notation, and why this is the identity tensor, let's form the dot product with the vector $\boldsymbol{\omega}$ on the right:

$$\begin{aligned}\mathbb{I}\cdot\boldsymbol{\omega} &= \hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\boldsymbol{\omega})+\hat{\mathbf{y}}(\hat{\mathbf{y}}\cdot\boldsymbol{\omega})+\hat{\mathbf{z}}(\hat{\mathbf{z}}\cdot\boldsymbol{\omega}) &(4.11)\\ &= \hat{\mathbf{x}}\,\omega_x+\hat{\mathbf{y}}\,\omega_y+\hat{\mathbf{z}}\,\omega_z &(4.12)\\ &= \boldsymbol{\omega}. &(4.13)\end{aligned}$$

We get the same result when we form the dot product on the left. When we take dot products with two $\boldsymbol{\omega}$'s, one on each side, we get a familiar scalar:

$$\boldsymbol{\omega}\cdot\mathbb{I}\cdot\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\boldsymbol{\omega}. \tag{4.14}$$

The identity tensor is not special to a specific choice of orthonormal basis, as the formula (4.10) might suggest. We could just as well have defined it as

$$\mathbb{I} = \hat{\mathbf{x}}'\hat{\mathbf{x}}'+\hat{\mathbf{y}}'\hat{\mathbf{y}}'+\hat{\mathbf{z}}'\hat{\mathbf{z}}' \tag{4.15}$$

for some other orthonormal basis. If you think of what is going on when you form a dot product with some arbitrary vector, as in $\mathbf{v}\cdot\mathbb{I}$, you will see why this is true. First the vector $\mathbf{v}$ is "analyzed" for its components with respect to the arbitrary basis. The components are then combined with the corresponding (arbitrary) basis vectors to give back the original vector.

We can express T_{rot} as a double $\boldsymbol{\omega}$ dot product with a tensor of order 2, the *moment of inertia tensor*:

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}, \tag{4.16}$$

$$\mathbf{I} = \sum_i m_i\left((\mathbf{r}_i\cdot\mathbf{r}_i)\mathbb{I}-\mathbf{r}_i\mathbf{r}_i\right). \tag{4.17}$$

The first term in $\mathbf{I}$ is just a scalar multiple of the identity tensor. All the complications in rigid body motion can be traced to the fact that, in general, $\mathbf{I}$ is no longer a multiple of the identity tensor when the second term is included.

In one of your homework problems, using many of the same manipulations that led to (4.16), you will show that the angular momentum of the rigid body has the form

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}. \tag{4.18}$$

From this we see that the two vectors, $\mathbf{L}$ and $\boldsymbol{\omega}$, are only parallel in the exceptional case where $\mathbf{I}$ is a multiple of the identity tensor. To help us identify those exceptional cases, and sort things out otherwise, we will write $\mathbf{I}$ in terms of a very special basis.

4.1.3 Principal axes

Let's start by writing $\mathbf{I}$ in terms of an arbitrary basis fixed in the body. Omitting the primes we normally reserve for body-fixed quantities, the positions of the masses are written as

$$\mathbf{r}_i = x_i \hat{\mathbf{x}} + y_i \hat{\mathbf{y}} + z_i \hat{\mathbf{z}}. \tag{4.19}$$

Substituting this into (4.17) we obtain

$$\mathbf{I} = I_{xx}\, \hat{\mathbf{x}}\hat{\mathbf{x}} + I_{xy}\, \hat{\mathbf{x}}\hat{\mathbf{y}} + \cdots, \tag{4.20}$$

where

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) \tag{4.21}$$

$$I_{xy} = \sum_i m_i (-x_i y_i), \tag{4.22}$$

and you should have no trouble writing down expressions for the seven other tensor components. Symmetry properties, such as $I_{xy} = I_{yx}$, follow from the fact that $\mathbf{I}$ is unchanged when the pairs of vectors in its construction are swapped.

To help us find a better basis we rewrite (4.20) using matrix notation:

$$\mathbf{I} = [\hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad \hat{\mathbf{z}}] \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix}. \tag{4.23}$$

Now consider a new basis, obtained from the original basis by the application of an orthogonal matrix U (a rotation):

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = U \begin{bmatrix} \hat{\mathbf{1}} \\ \hat{\mathbf{2}} \\ \hat{\mathbf{3}} \end{bmatrix}, \qquad [\hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad \hat{\mathbf{z}}] = [\hat{\mathbf{1}} \quad \hat{\mathbf{2}} \quad \hat{\mathbf{3}}]\, U^T. \tag{4.24}$$

The idea behind the curious numerical names for the transformed orthonormal basis vectors will be clear later. Substituting these expressions into (4.23) we obtain the following matrix of tensor components in the transformed basis:

$$U^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} U. \tag{4.25}$$

Because the middle matrix is symmetric, there exists a particular transformation U that makes the resulting matrix diagonal:

$$U^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} U = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}. \tag{4.26}$$

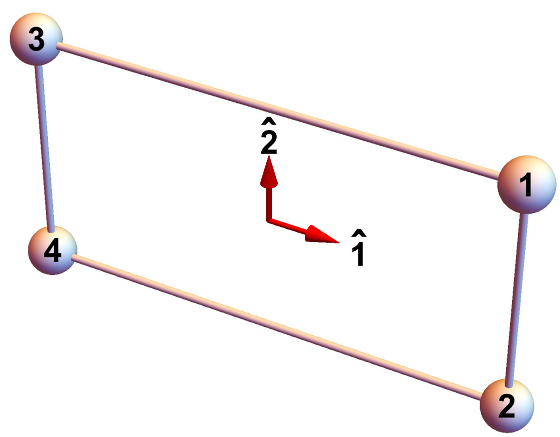
The diagonal elements are called the *principal moments of inertia* of the rigid body. In this special transformed basis, called the *principal basis*, the moment of inertia tensor takes a very simple form:

$$\mathbf{I} = I_1\,\hat{\mathbf{1}}\hat{\mathbf{1}} + I_2\,\hat{\mathbf{2}}\hat{\mathbf{2}} + I_3\,\hat{\mathbf{3}}\hat{\mathbf{3}}. \tag{4.27}$$

From this you can see that a condition for $\mathbf{I}$ to be a multiple of the identity tensor is equality of all three principal moments of inertia. The axes in the body defined by the basis vectors $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ are called the *principal axes*.

When a body has symmetry, the principal axes are symmetry axes of the body. As an example, consider four equal masses m arranged at the corners of a rectangle of side length $2a$ and $2b$. Choosing a basis where two of the vectors are parallel to the rectangle edges, the positions of the four masses are:

$$\mathbf{r}_1 = a\,\hat{\mathbf{1}} + b\,\hat{\mathbf{2}} \qquad \mathbf{r}_2 = a\,\hat{\mathbf{1}} - b\,\hat{\mathbf{2}} \qquad \mathbf{r}_3 = -a\,\hat{\mathbf{1}} + b\,\hat{\mathbf{2}} \qquad \mathbf{r}_4 = -a\,\hat{\mathbf{1}} - b\,\hat{\mathbf{2}}. \tag{4.28}$$



Using these in the original tensor expression (4.17), we find

$$\sum_i (\mathbf{r}_i \cdot \mathbf{r}_i)\mathbb{I} = 4(a^2 + b^2)(\hat{\mathbf{1}}\hat{\mathbf{1}} + \hat{\mathbf{2}}\hat{\mathbf{2}} + \hat{\mathbf{3}}\hat{\mathbf{3}}) \tag{4.29}$$

$$\sum_i \mathbf{r}_i\mathbf{r}_i = (a\,\hat{\mathbf{1}} + b\,\hat{\mathbf{2}})(a\,\hat{\mathbf{1}} + b\,\hat{\mathbf{2}}) + (a\,\hat{\mathbf{1}} - b\,\hat{\mathbf{2}})(a\,\hat{\mathbf{1}} - b\,\hat{\mathbf{2}}) + \cdots \tag{4.30}$$

$$= 4a^2\,\hat{\mathbf{1}}\hat{\mathbf{1}} + 4b^2\,\hat{\mathbf{2}}\hat{\mathbf{2}}. \tag{4.31}$$

Subtracting the two parts we obtain a moment of inertia tensor of the form (4.27):

$$\mathbf{I} = m\left(4b^2\,\hat{\mathbf{1}}\hat{\mathbf{1}} + 4a^2\,\hat{\mathbf{2}}\hat{\mathbf{2}} + 4(a^2 + b^2)\hat{\mathbf{3}}\hat{\mathbf{3}}\right). \tag{4.32}$$