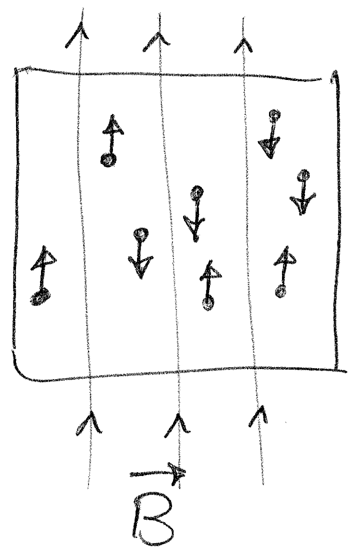
Lecture 39

system of N spin-½ magnetic moments



$$E = -\vec{\mu}_1 \cdot \vec{B} - \cdots - \vec{\mu}_N \cdot \vec{B}$$

spin-½ : $\vec{\mu}_i \cdot \vec{B} = \pm b$

$\uparrow$ energy $= -b$ "up"

$\downarrow$ energy $= +b$ "down"

n = num. flipped spins

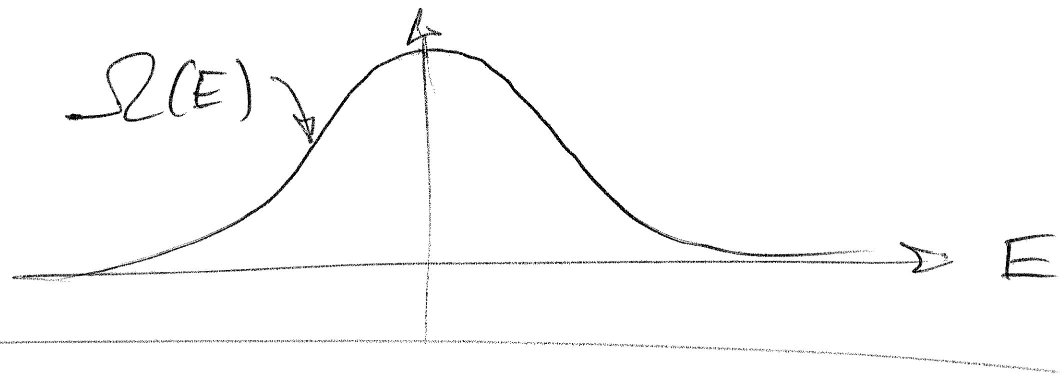
$n=0$	$\uparrow\uparrow \cdots \uparrow$	$E = -Nb$	1 state
$n=1$	$\downarrow\uparrow \cdots \uparrow$	$E = (-N+2)b$	N states
$n=2$	$\downarrow\downarrow\uparrow \cdots \uparrow$	$E = (-N+4)b$	$\frac{N(N-1)}{2}$ states
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n=N$	$\downarrow\downarrow \cdots \downarrow$	$E = +Nb$	1 state

$$E = (-N + 2n)b$$

Assuming N = even, we have the most microstates when $n = N/2$ (equal numbers up and down) and $E = 0$.

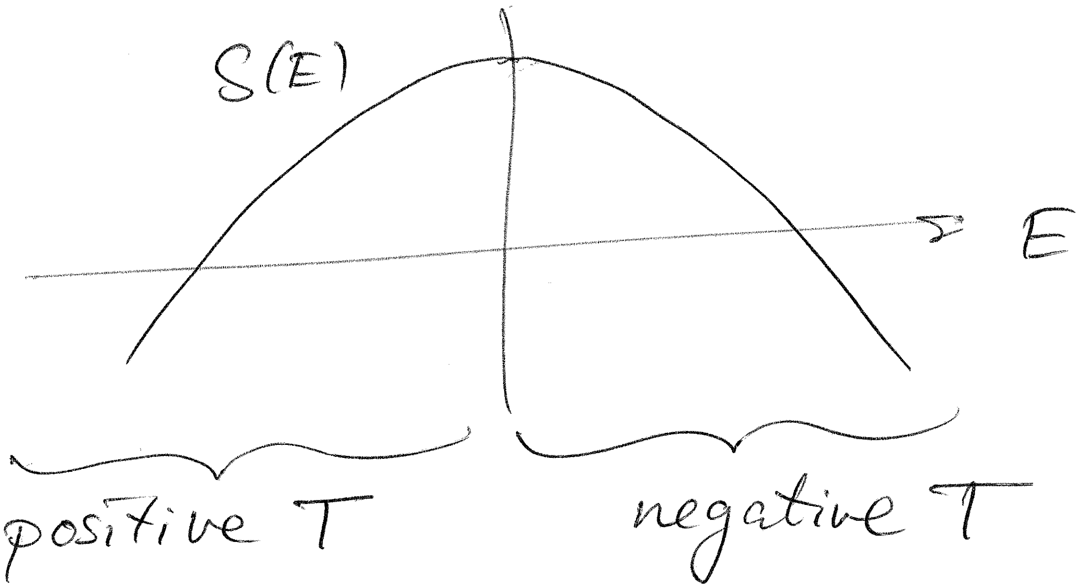
$$\Omega(E) = \begin{pmatrix} \text{num. microstates} \\ \text{with } E = (-N+2n)b \\ (n \text{ flipped}) \end{pmatrix} = \frac{N!}{n!(N-n)!}$$

$$\underset{\text{large } N}{\simeq} \underbrace{2^N \sqrt{\frac{2}{\pi N}}}_{\text{constant}} \; \underbrace{e^{-\frac{2}{N}(n - N/2)^2}}_{e^{-\frac{2}{N}(E/2b)^2}} \; *$$



* Derive using Stirling's formula for factorials

$$S(E) = S_0 - k_B \frac{2}{N} \left(E/2b\right)^2$$



As this system is heated so its energy increases from just below $E=0$ to just above $E=0$, its temperature goes from $+\infty$ to $-\infty$!

$$\frac{1}{T} = \frac{dS}{dE} = -k_B \frac{E/b^2}{N}$$

$$\Rightarrow \quad E = -\frac{b^2}{k_B T} N$$

(not proportional to $k_B T$ as in gases, solids)

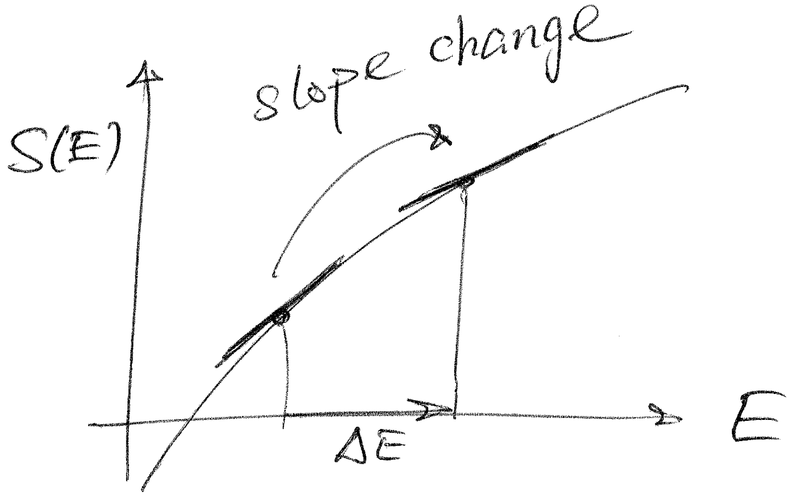
<u>thermal</u> <u>reservoir</u>:

a system whose temperature is very insensitive to a (modest) change in energy (added or extracted as heat)

We will see that "temperature insensitivity" is closely linked to the <u>number</u> of <u>degrees of freedom</u> of the system, also called the <u>heat</u> <u>capacity</u>.

The temperature of a system changes with energy because the slope of $S(E)$ varies with E:

39.5



$$\frac{dS}{dE} = \frac{1}{T(E)}$$

heat capacity, C, is the inverse of the temperature sensitivity:

$$C = \left(\frac{dT(E)}{dE}\right)^{-1} = \frac{\Delta E}{\Delta T}$$

Take another E derivative of S:

$$\frac{d}{dE}\frac{dS}{dE} = \frac{d}{dE}\left(\frac{1}{T(E)}\right) = -\frac{1}{T^2}\frac{dT}{dE}$$

$$\Rightarrow C = \frac{dE}{dT} = -\frac{1}{T^2}\left(\frac{d^2S}{dE^2}\right)^{-1}$$

Let's work out the heat capacity of a solid of N atoms: (39.6)

$$S = 3k_B N \log E + S_0$$

$$\frac{dS}{dE} = 3k_B \frac{N}{E} = \frac{1}{T(E)}$$

$$T(E) = \frac{E}{3k_B N}, \quad \frac{dT}{dE} = \frac{1}{3k_B N}$$

$$C = \frac{dE}{dT} = 3k_B N$$

The "capacity" is proportional to the number of atoms or, more generally, the number of degrees of freedom. An "ideal" reservoir has an infinite number of degrees of freedom, so no finite amount of heat energy will change its temperature.

Now consider a microscopic system in contact with a thermal reservoir at temperature T_R. Suppose the microscopic system is just a single atom in a gas, for example, and the reservoir is all the other atoms.

By keeping track of which micro-states $i = 1, 2, ..$ the atom is in, we'll know how much energy the reservoir has and from that how many states it can access:

state of atom	atom energy	reservoir energy	total microstates
$i=1$	E_1	$E-E_1$	$\Omega_R(E-E_1)$
2	E_2	$E-E_2$	$\Omega_R(E-E_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\Omega_R(E-E_i) = \begin{pmatrix}\text{number of states} \\ \text{when atom is in} \\ \text{state } i\end{pmatrix} \propto \begin{pmatrix}\text{probability atom} \\ \text{is in state } i\end{pmatrix}$$

$$\| $$

$$P_i$$

Since $E_i \ll E$ (definition of a microscopic system (atom, etc.)) we can approximate Ω_R:

$$\Omega_R(E-E_i) = e^{S_R(E-E_i)/k_B}$$

$$S_R(E-E_i) \simeq S_R(E) - E_i \underbrace{\left.\frac{dS}{dE}\right|_E}_{T_R^{-1}}$$

$$\Omega_R(E-E_i) \simeq e^{S_R(E)/k_B - E_i/k_B T_R}$$

independent of atom state i

$$\Rightarrow P_i \propto e^{-E_i/k_B T_R}$$

This is the "Boltzmann probability" that an atom (or other microscopic system) is in a state whose energy is E_i.