Lecture 39: May 10

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39.1 Geometric action principles and relativity (continued)

39.1.1 Symmetries of the string-wave equation

In lecture 38 we observed that the Lagrangian for the world-surface

$$s^\alpha(x,t) = (ct\,, x\,, y(x,t)\,, 0) \tag{39.1}$$

for a nearly straight string parameterized by x and t reduced to the Lagrangian of the simple elastic string in the small y-amplitude limit. And because the action was defined as a Lorentz-invariant — the surface area in Minkowski space-time — it was not surprising to find that the velocity of small amplitude wave propagation was exactly the speed of light c.

In a homework assignment you were asked to obtain the equations of motion for the world-surface (39.1) without making the small amplitude approximation. The resulting wave equation now includes cubic terms:

$$\partial_x^2 y - \partial_t^2 y = (\partial_x y)^2 \partial_t^2 y - 2(\partial_x y)(\partial_t y)\partial_x\partial_t y + (\partial_t y)^2 \partial_x^2 y. \tag{39.2}$$

To minimize clutter, in this equation and all remaining equations in this lecture, we have set $c = 1$ (in effect using the same units for space and time). In the linear (small amplitude) approximation,

$$\partial_x^2 y - \partial_t^2 y \approx 0, \tag{39.3}$$

this equation has many symmetries, including translation in space and time and Lorentz-transformations for relative motion along x.

One symmetry that this equation does *not* have is rotational symmetry. For suppose we have an initial string amplitude $y(x, 0)$ and velocity $\partial_t y(x, 0)$ and time-evolved these with (39.3). If the equation possessed rotational symmetry then we should be able to use the rotated-by-θ initial amplitude

$$y'(x,0) = y(x,0)\cos\theta + x\sin\theta, \tag{39.4}$$

and a likewise rotated initial velocity, then time-evolve with (39.3) and thereby obtain a rotated version of the original time evolution. But we know this cannot possibly be correct because (39.4) is *not* small-amplitude unless $\theta = 0$. On the other hand, we know the nonlinear equation (39.2) will give the correctly rotated time evolution because it is derived from a geometric action that is invariant — among other things — under rotations of the (x, y) plane. The fact that x and y have such different roles in the nonlinear equation — independent vs. dependent variable — should not bother you. This is a casualty of using a particular spatial axis, in this case x, for one of the arbitrary world-surface parameters.

39.1.2 General solution of the equations for the relativistic string

Non-linearity, in particular the cubic terms in (39.2), seems like a high price to pay in order to have equations that respect all the symmetries of a mechanical system! Fortunately, in the case of the relativistic string, these non-linearities do not present an obstacle to constructing the most general solution.

The string action (without the scale factor),

$$S[s] = \int \mathcal{L}\, dp\, dq, \tag{39.5}$$

$$\mathcal{L} = \sqrt{(\partial_p s^\alpha\, \partial_q s_\alpha)^2 - (\partial_p s^\alpha\, \partial_p s_\alpha)(\partial_q s^\beta\, \partial_q s_\beta)}, \tag{39.6}$$

now in complete generality, suggests a possible simplification. Exercising our freedom to choose a parameterization, we get an enormous simplification when both tangent vectors are always null:

$$\partial_p s^\alpha\, \partial_p s_\alpha = 0 \qquad \partial_q s^\alpha\, \partial_q s_\alpha = 0. \tag{39.7}$$

Supposing there exists such a parameterization, the action would reduce to the following:

$$S[s] = \int \partial_p s^\alpha\, \partial_q s_\alpha\, dp\, dq. \tag{39.8}$$

However, before we derive equations of motion for this action, we need to ask how Hamilton's principle is affected by the constraints (39.7). Consider perturbing an arbitrary world-surface $s^\star$ that has a "null-parameterization":

$$s^\alpha(p,q) = s^{\star\alpha}(p,q) + \delta s^\alpha(p,q). \tag{39.9}$$

Under this perturbation we have

$$\begin{aligned} (\partial_p s^\alpha\, \partial_p s_\alpha)(\partial_q s^\beta\, \partial_q s_\beta) &= 4(\partial_p s^{\star\alpha}\, \partial_p \delta s_\alpha)(\partial_q s^{\star\beta}\, \partial_q \delta s_\beta) + O\left((\delta s)^3\right) && (39.10)\\ &= O\left((\delta s)^2\right), && (39.11)\end{aligned}$$

whereas

$$\begin{aligned} (\partial_p s^\alpha\, \partial_q s_\alpha)^2 &= (\partial_p s^{\star\alpha}\, \partial_q s^\star{}_\alpha + \partial_p s^{\star\alpha}\, \partial_q \delta s_\alpha + \partial_p \delta s^\alpha\, \partial_q s^\star{}_\alpha + \cdots)^2 && (39.12)\\ &= (\partial_p s^{\star\alpha}\, \partial_q s^\star{}_\alpha)^2 + O\left(\delta s\right). && (39.13)\end{aligned}$$

The terms (39.10) missing from the simplified action (39.8) therefore only add terms of order $(\delta s)^2$ under the perturbation and imply we can impose the usual order $(\delta s)^1$ extremal condition to (39.8).

The equations of motion

$$0 = \partial_p \left(\frac{\partial \mathcal{L}'}{\partial(\partial_p s_\alpha)} \right) + \partial_q \left(\frac{\partial \mathcal{L}'}{\partial(\partial_q s_\alpha)} \right) \tag{39.14}$$

for the simplified Lagrangian

$$\mathcal{L}' = \partial_p s^\alpha\, \partial_q s_\alpha \tag{39.15}$$

are very simple:

$$0 = \partial_p \partial_q s^\alpha + \partial_q \partial_p s^\alpha = 2\partial_p \partial_q s^\alpha\,. \tag{39.16}$$

The most general solution is given by

$$s^\alpha(p,q) = f^\alpha(p) + g^\alpha(q), \tag{39.17}$$

where by (39.7) the world-lines $f(p)$ and $g(q)$ are arbitrary null curves:

$$\begin{aligned} \partial_p f^\alpha\, \partial_p f_\alpha &= 0 && (39.18)\\ \partial_q g^\alpha\, \partial_q g_\alpha &= 0. && (39.19)\end{aligned}$$

39.1.3 Colliding waves on a relativistic string

A convenient way to parameterize general null curves is in terms of "particles" that are always moving with the speed of light — though not necessarily along straight lines:

$$f^\alpha(p) = (p\,, \mathbf{f}(p)) \tag{39.20}$$
$$g^\alpha(q) = (q\,, \mathbf{g}(q)). \tag{39.21}$$

Here $\mathbf{f}(p)$ and $\mathbf{g}(q)$ are general trajectories in space of particles moving at light-speed:

$$\dot{\mathbf{f}} = \mathbf{u}, \quad |\mathbf{u}| = 1, \tag{39.22}$$
$$\dot{\mathbf{g}} = \mathbf{v}, \quad |\mathbf{v}| = 1. \tag{39.23}$$

Switching to parameters $r = p - q$ and $t = p + q$ in the solution (39.17), we obtain the following expression for the world-surface:

$$s^0(r,t) = t, \tag{39.24}$$
$$\mathbf{s}(r,t) = \mathbf{f}\left(\frac{t+r}{2}\right) + \mathbf{g}\left(\frac{t-r}{2}\right). \tag{39.25}$$

The parameter t is therefore the time at which the string is observed and $\mathbf{s}(r,t)$ is the space curve traced out by the string, at any t, as a function of a parameter r along its length.

To see evidence of non-linear characteristics in our sum-of-null-curves general solution, we consider a wave scattering scenario. On an asymptotically straight string we place two wave forms moving in opposite directions. Choosing the asymptotic string along the direction $\hat{\mathbf{x}}$, we require

$$\lim_{r\to\pm\infty} \mathbf{f}\left(\frac{t+r}{2}\right) + \mathbf{g}\left(\frac{t-r}{2}\right) = Cr\,\hat{\mathbf{x}}, \tag{39.26}$$

where C is a constant. Taking the r-derivative of both sides,

$$\lim_{r\to\pm\infty} \frac{1}{2}\mathbf{u}\left(\frac{t+r}{2}\right) - \frac{1}{2}\mathbf{v}\left(\frac{t-r}{2}\right) = C\,\hat{\mathbf{x}}, \tag{39.27}$$

we see there is a solution of the type we seek when the velocity $\mathbf{u}$ of one of the particles approaches $+\hat{\mathbf{x}}$ in the distant past and future, while $\mathbf{v}$ of the other particle approaches $-\hat{\mathbf{x}}$ in the same limits (and $C = 1$).

When the 3-velocities are constant, in particular $\mathbf{u}(p) = +\hat{\mathbf{x}}$ and $\mathbf{v}(p) = -\hat{\mathbf{x}}$, then the string is always straight and there are no waves. Deviating from these constant velocities over a finite parameter range (respectively in p and q) produces finite wave-forms $\mathbf{f}$ and $\mathbf{g}$ (upon integrating respectively $\mathbf{u}$ and $\mathbf{v}$) that move in opposite directions by (39.25). Does anything interesting happen when the wave amplitudes are sufficiently large?

For insights on the collision process, let's express the action in terms of the null curves:

$$S[s] = \int \partial_p s^\alpha\, \partial_q s_\alpha\, dp\, dq \tag{39.28}$$
$$= \int \partial_p f^\alpha\, \partial_q g_\alpha\, dp\, dq \tag{39.29}$$
$$= \int (-1 + \dot{\mathbf{f}}\cdot\dot{\mathbf{g}}) dp\, dq \tag{39.30}$$
$$= \int (-1 + \mathbf{u}\cdot\mathbf{v}) dp\, dq. \tag{39.31}$$

The vanishing of the Lagrangian, as we saw in lecture 35, corresponds to the string's transverse velocity having reached light-speed. This is as singular as a relativistic string can get, say as exhibited by the oscillating circular string (lecture 36) at periodic instants of time. For this to happen with the colliding wave forms, the 3-velocity $\mathbf{u}$ must deviate sufficiently from $+\hat{\mathbf{x}}$ (its asymptotic value), and $\mathbf{v}$ must deviate sufficiently from $-\hat{\mathbf{x}}$, so that the condition $\mathbf{u} \cdot \mathbf{v} = 1$ is achieved somewhere on the world-surface. This in turn implies a minimum amplitude for the wave forms $\mathbf{f}$ and $\mathbf{g}$.

Drawing exercise: Sketch a pair of velocity curves $\mathbf{u}(p)$ and $\mathbf{v}(q)$ on the unit sphere that have the asymptotic behavior discussed and also meet the condition $\mathbf{u}(p) \cdot \mathbf{v}(q) = 1$ for particular parameter combinations.