Physics 3318: Analytical Mechanics	Spring 2017

Lecture 38: May 8

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38.1 Geometric action principles and relativity (continued)

38.1.1 Small amplitude limit of the relativistic string

We remarked in lecture 36 that the uniformly moving straight string — whose world surface is a flat plane — is a solution of the non-linear equations for the relativistic string. When the string is slightly perturbed from this simple solution, the equation for the perturbation will be a linear equation. Do we recover the familiar, non-relativistic equations for the simple elastic string?

To simplify the analysis we perturb the stationary ($v = 0$) case of the straight string of lecture 36 with only a transverse y amplitude. Using time t and position x along the string as parameters, the world-surface has the form

$$s^\alpha(x,t) = (ct, x, y(x,t), 0), \tag{38.1}$$

where $y(x,t)$ is an arbitrary function but is treated as a perturbation. From the tangent vectors

$$\begin{aligned} \partial_x s^\alpha &= (0\,, 1\,, \partial_x y\,, 0) && (38.2)\\ \partial_t s^\alpha &= (c\,, 0\,, \partial_t y\,, 0), && (38.3) \end{aligned}$$

we obtain the following for the Lagrangian:

$$\begin{aligned} \mathcal{L}^2 &= -a^{\alpha\beta}a_{\alpha\beta} && (38.4)\\ &= (\partial_x s^\alpha\,\partial_t s_\alpha)^2 - (\partial_x s^\alpha\,\partial_x s_\alpha)(\partial_t s^\beta\,\partial_t s_\beta) && (38.5)\\ &= (\partial_x y)^2(\partial_t y)^2 - \left(1 + (\partial_x y)^2\right)\left(-c^2 + (\partial_t y)^2\right) && (38.6)\\ &= c^2\left(1 - \frac{1}{c^2}(\partial_t y)^2 + (\partial_x y)^2\right), && (38.7) \end{aligned}$$

$$\mathcal{L} = c - \frac{1}{2c}(\partial_t y)^2 + \frac{c}{2}(\partial_x y)^2 + O(y^4). \tag{38.8}$$

Let's compare the action for this geometric Lagrangian, after including the scale parameter C, with the action for the simple elastic string from lecture 13:

$$\begin{aligned} S_{\text{geom}}[y] &= C\int\left(c - \frac{1}{2c}(\partial_t y)^2 + \frac{c}{2}(\partial_x y)^2\right)dx\,dt && (38.9)\\ S_{\text{elastic}}[y] &= \int\left(\frac{\mu}{2}(\partial_t y)^2 - \frac{\tau}{2}(\partial_x y)^2\right)dx\,dt. && (38.10) \end{aligned}$$

Up to an irrelevant additive constant, and a multiplicative constant that also does not affect the equations of motion, they have the same form. For the elastic action to be consistent with the geometric action we must have the following relation between the mass per unit length μ and the tension τ:

$$c\mu = \tau/c. \tag{38.11}$$

Not surprisingly, an elastic string with these characteristics would have wave speed $\sqrt{\tau/\mu} = c$.

Our best strategy for giving the scale factor C of the relativistic string a physical interpretation is to use the stress tensor construction of lecture 37 to calculate the string's energy and momentum. We'll continue to treat the transverse amplitude y as a perturbation so we can make contact with the simple elastic string.

Repeating the derivation of the stress tensor in lecture 37, now for strings, we begin with the vector density

$$0 = C\int -\partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a r_\alpha)}\right)\,\delta^4(x - s(p))\,dp\,dq, \tag{38.12}$$

which is identically zero by the string's Euler-Lagrange equations. The subsequent steps of the derivation are nearly the same and and show that this vector density is the 4-divergence of

$$T^{\alpha\beta}(x) = C\int(\mathcal{L}\,dp\,dq)\,v^{\alpha\gamma}v_\gamma{}^\beta\,\delta^4\left(x - s(p,q)\right), \tag{38.13}$$

where $v^{\alpha\beta}$ is the normalized anti-symmetric surface element defined in lecture 36, analogous to the normalized line element u^α of world-lines. In a homework assignment you are asked to show directly that

$$\partial_\beta T^{\alpha\beta} = 0 \tag{38.14}$$

follows from definition (38.13) and the equations of motion.

As we saw in lecture 37, the vanishing divergence leads by Stokes' theorem to the definition of a conserved 4-momentum:

$$\begin{aligned}
P^\alpha(t') &= \int d^3x'\,T^{\alpha 0}(ct',\mathbf{x}') && (38.15)\\
&= C\int d^3x'\int(\mathcal{L}\,dx\,dt)\,v^{\alpha\gamma}v_\gamma{}^0\,\delta\left(ct' - s^0(x,t)\right)\delta^3\left(\mathbf{x}' - \mathbf{s}(x,t)\right) && (38.16)\\
&= C\int d^3x'\int(\mathcal{L}\,dx\,dt)\,v^{\alpha\gamma}v_\gamma{}^0\,\delta\left(ct' - ct\right)\delta^3\left(\mathbf{x}' - \mathbf{s}(x,t)\right) && (38.17)\\
&= (C/c)\int(\mathcal{L}\,dx)\,v^{\alpha\gamma}v_\gamma{}^0 && (38.18)\\
&= (C/c)\int\left(a^{\alpha\gamma}a_\gamma{}^0/\mathcal{L}\right)dx. && (38.19)
\end{aligned}$$

We have used the (x,t) parameterization of the string, for which $s^0(x,t) = ct$. The integrand of the last line corresponds to an energy-momentum density per unit length that we will evaluate for our nearly straight string.

From the tangent vectors (38.2) and (38.3) we obtain

$$a^{\alpha\gamma}a_\gamma{}^0 = \frac{1}{2}\begin{bmatrix} 0 & -c & -c\,\partial_x y & 0\\ c & 0 & \partial_t y & 0\\ c\,\partial_x y & -\partial_t y & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 0\\ c\\ c\,\partial_x y\\ 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix} -c^2(1+(\partial_x y)^2)\\ +c(\partial_t y)(\partial_x y)\\ -c(\partial_t y)\\ 0\end{bmatrix} \tag{38.20}$$

and from (38.8)

$$c/\mathcal{L} = 1 + \frac{1}{2c^2}(\partial_t y)^2 - \frac{1}{2}(\partial_x y)^2 + O(y^4). \tag{38.21}$$

For the energy we then have (for $\alpha = 0$)

$$\begin{aligned} E = cP^0 &= -\frac{C}{2}\int c\left(1+(\partial_x y)^2\right)\left(1 + \frac{1}{2c^2}(\partial_t y)^2 - \frac{1}{2}(\partial_x y)^2 + O(y^4)\right)dx & (38.22)\\ &= -\frac{C}{2}\int c\left(1 + \frac{1}{2c^2}(\partial_t y)^2 + \frac{1}{2}(\partial_x y)^2 + O(y^4)\right)dx. & (38.23)\end{aligned}$$

In order for the second term to match the kinetic energy of the (non-relativistic) elastic string we must have

$$-\frac{C}{2} = \mu c. \tag{38.24}$$

The first term is then exactly the rest energy of a string with mass density μ.

By (38.20) the momentum along y ($\alpha = 2$) is the dominant component for small perturbations:

$$\begin{aligned} p_y = P^2 &= -\frac{C}{2}\int \frac{1}{c}\partial_t y\left(1 + \frac{1}{2c^2}(\partial_t y)^2 - \frac{1}{2}(\partial_x y)^2 + O(y^4)\right)dx & (38.25)\\ &= -\frac{C}{2}\int \frac{1}{c}\left(\partial_t y + O(y^3)\right). & (38.26)\end{aligned}$$

This too exactly matches the momentum of the elastic string when we use (38.24).