Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 37: May 5

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37.1 Geometric action principles and relativity (continued)

37.1.1 Relativistic energy-momentum conservation

In non-relativistic mechanics we saw that conservation laws are consequences of continuous symmetries of the system. We were able to define a conserved energy whenever the Lagrangian had no direct dependence on time: the system Lagrangian was then invariant under arbitrary time-translations. Likewise, in systems where the Cartesian coordinates are the degrees of freedom, the absence of these in the Lagrangian implies that their conjugate momenta are conserved.

Revising the definitions of conservation laws to be consistent with special relativity presents some challenges. The definitions of conjugate momentum and the Hamiltonian rely on the distinguished role of time in an essential way — something that must be abandoned in a relativistic definition. At an even more basic level, the very notion of "conservation law" has problems in a relativistic setting. The statement that the energy of the system "before" is the same as it is "after" assumes a particular slicing of space-time into events that are simultaneous at one time (before) and another (after). Different inertial observers will not agree on what these slices of simultaneity should be, and a relativistic conservation law should reflect this ambiguity.

We will use the conservation of charge — a scalar quantity — to motivate a geometric and relativistically consistent definition of energy-momentum. Gauss' law relates the charge enclosed by a surface to the flux of electric field through the surface. Suppose we have two concentric spheres, the innermost enclosing some charge and no charges elsewhere in space. The flux through the inner sphere exactly matches the flux through the outer sphere because there are no sources of flux between the spheres. Flux is "conserved" if we interpret the inner sphere as "before" and the outer sphere as "after".

Just as evidence of a conserved scalar charge Q is gained (in three dimensions) from the flux of the 3-vector electric field $\mathbf{E}$ it generates, the four components of the energy-momentum 4-vector P^α can be inferred (in four dimensions) from the flux of an entity $T^{\alpha\beta}(x)$ with an additional index. We will construct such an entity for particles and strings called the *energy-momentum stress tensor density*[1]. Just as "conservation" of electric field ($\mathbf{E}$) flux is implied by the vanishing of $\nabla\cdot\mathbf{E}$ — a local property, the conservation of stress-energy ($T^{\alpha\beta}$) flux will follow from the vanishing 4-divergence

$$\partial_\beta T^{\alpha\beta} = 0. \tag{37.1}$$

This local property, which avoids simultaneity/reference-frame ambiguity, is the more fundamental statement of conservation; the more familiar conservation of energy-momentum is its most important consequence.

[1]The term "density" is often left out, because this tensor is only defined in that particular context.

37.1.2 Energy-momentum stress tensor for particles

As we saw in the Noether construction of conserved quantities (lecture 17), the equations of motion play a central role, as do continuous symmetries of the system Lagrangian. Recall the Lagrangian for a relativistic particle:

$$\begin{aligned} S[r] &= Mc\int\sqrt{-(\partial_p r^\alpha)(\partial_p r_\alpha)}\,dp && (37.2)\\ &= Mc\int \mathcal{L}(\partial_p r)\,dp. && (37.3)\end{aligned}$$

This Lagrangian is invariant under arbitrary translations of the system variables, $r^\alpha(p) \to r^\alpha(p) + a^\alpha$, because it only depends on the p-derivatives of r. The Euler-Lagrange equations therefore lack the first (non-derivative) term:

$$0 = 0 - \partial_p\left(\frac{\partial\mathcal{L}}{\partial(\partial_p r_\alpha)}\right). \tag{37.4}$$

Now suppose we "attach" to all events of the world-line $r(p)$ a delta-function density of zero-weight, where "zero" is defined by the equations of motion (37.4) being satisfied (at all events along the world-line):

$$0 = \int -\partial_p\left(\frac{\partial\mathcal{L}}{\partial(\partial_p r_\alpha)}\right)\,\delta^4(x - r(p))\,dp. \tag{37.5}$$

This is a vector (upper index α) density (argument x) that happens to be identically zero by the equations of motion. Performing an integration by parts we obtain

$$\begin{aligned} 0 &= \int \frac{\partial\mathcal{L}}{\partial(\partial_p r_\alpha)}\,\partial_p\delta^4(x - r(p))\,dp && (37.6)\\ &= \int -\frac{\partial\mathcal{L}}{\partial(\partial_p r_\alpha)}\,(\partial_p r^\beta)\,\partial_\beta\delta^4(x - r(p))\,dp, && (37.7)\end{aligned}$$

where we made use of an identity from lecture 34. Before we proceed, we should examine the boundary terms we omitted in the integration by parts:

$$\frac{\partial\mathcal{L}}{\partial(\partial_p r_\alpha)}\,\delta^4(x - r(p))\bigg|_{p_1}^{p_2} \tag{37.8}$$

This vanishes provided we make x avoid the end-events of the world-line. A more physical statement of the same thing is that our derivation only applies to stress-energy conservation at times x^0/c between the times $r^0(p_1)/c$ and $r^0(p_2)/c$ where the motion starts and stops (by unspecified mechanisms).

Note that (37.7) is a divergence, and in fact a divergence that vanishes (up to the endpoint restrictions). Restoring the scale constant Mc, this defines the energy-momentum stress tensor density

$$\begin{aligned} T^{\alpha\beta}(x) &= Mc\int -\frac{\partial\mathcal{L}}{\partial(\partial_p r_\alpha)}\,(\partial_p r^\beta)\delta^4(x - r(p))\,dp && (37.9)\\ &= Mc\int\left(\frac{\partial_p r^\alpha}{\mathcal{L}}\right)(\partial_p r^\beta)\delta^4(x - r(p))\,dp && (37.10)\\ &= Mc\int u^\alpha u^\beta\,\delta^4(x - r(p))\,(\mathcal{L}dp). && (37.11)\end{aligned}$$

Since $\mathcal{L}dp$ is a length (c times proper time), and the 4-velocity u^α is dimensionless, $T^{\alpha\beta}(x)$ is a momentum 3-density: when integrated over a 3-volume it produces a momentum.

37.1.3 Energy-momentum conservation

Stokes' theorem applied to the integral of the 4-divergence (37.1) over a 4-volume V with boundary ∂V is the statement,

$$0 = \int_V \partial_\beta T^{\alpha\beta}\, d^4x = \int_{\partial V} T^{\alpha\beta} n_\beta\, d^3x. \tag{37.12}$$

where n_α is the outward-directed unit normal to ∂V. Let's evaluate the surface integral for a space-time 4-cylinder comprising "before" and "after" ends — solid 3-spheres at times t_1 and t_2 — and a matching spatial 2-sphere evolved between times t_1 and t_2. We will make the sphere sufficiently large that the particle never strays outside V on the "spatial sides". The only places on the boundary ∂V where $T^{\alpha\beta}$ does not vanish is the two fixed-time ends with normals

$$n_\alpha(t_1) = (-1, 0, 0, 0), \qquad n_\alpha(t_2) = (+1, 0, 0, 0) \tag{37.13}$$

at the early (t_1) and late (t_2) times. Equation (37.12) then becomes

$$0 = P^\alpha(t_2) - P^\alpha(t_1), \tag{37.14}$$

where

$$\begin{aligned} P^\alpha(t_1) &= \int T^{\alpha 0}(ct_1, \mathbf{x})\, d^3x && (37.15)\\ P^\alpha(t_2) &= \int T^{\alpha 0}(ct_2, \mathbf{x})\, d^3x, && (37.16)\end{aligned}$$

and the space integrals are over the large solid 3-sphere. Although the tensor $T^{\alpha\beta}$ is symmetric, in this setting the first index is associated with the entity that is "flowing" and the second index defines the normal 3-surface through which the flux is being evaluated. We get a conventional conservation law when the latter index is 0, because by (37.14) there is no change in the flux of $T^{\alpha 0}$ between the two times.

Using the standard parameterization by time of the particle world line (lecture 36), we can evaluate a particle's conserved stress-energy flux:

$$\begin{aligned} P^\alpha(t_1) &= Mc \int d^3x \int (\mathcal{L}dt)\, u^\alpha u^0\, \delta(ct_1 - r^0(t))\, \delta^3(\mathbf{x} - \mathbf{r}(t)) && (37.17)\\ &= Mc \int (\mathcal{L}dt)\, u^\alpha u^0\, \delta(ct_1 - r^0(t)) && (37.18)\\ &= Mc\,(c/\gamma) u^\alpha \gamma \int dt\, \delta(ct_1 - ct) && (37.19)\\ &= Mc\, u^\alpha. && (37.20)\end{aligned}$$

As promised, this is the standard energy-momentum 4-vector of a particle. While there are less sophisticated methods to derive the relativistic counterparts of energy and momentum, we will need this general construction for calculating the energy and momentum of relativistic strings.