Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 36: May 3

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

36.1 Geometric action principles and relativity (continued)

36.1.1 Equations of motion

Recall the action and Lagrangian we defined for the relativistic particle:

$$S[r] = Mc\int \mathcal{L}\,dp, \tag{36.1}$$

$$\mathcal{L}(r, \partial_p r) = \sqrt{-(\partial_p r^\alpha)(\partial_p r_\alpha)}. \tag{36.2}$$

Since derivatives of the coordinates r are not with respect to time, as in ordinary mechanics, it is not immediately obvious that the equations of motion for r are the Euler-Lagrange equations applied to $\mathcal{L}$. But the Euler-Lagrange equations are the consequences of something deeper: Hamilton's principle. Therefore, a better question to ask is whether the physical basis of Hamilton's principle should continue to apply to the relativistic particle action.

The physical basis of Hamilton's principle is a property of quantum amplitudes. In particular, the dominant contribution to the amplitude sum in quantum mechanics can be traced to the constructive interference of amplitudes from trajectories that are close to the classical trajectory — for which the action is extremal. This principle makes no reference to time, nor does it single out time as the proper way to parametrize motion (a world-line). It thus seems reasonable to continue to apply the principle, now to an action that is explicitly Lorentz-invariant. When we do this, here is what the Euler-Lagrange equations give us:

$$0 = \frac{\partial \mathcal{L}}{\partial r^\alpha} - \partial_p\left(\frac{\partial \mathcal{L}}{\partial(\partial_p r^\alpha)}\right), \qquad \alpha = 0,1,2,3 \tag{36.3}$$

$$= \partial_p\left(\frac{\partial_p r_\alpha}{\mathcal{L}}\right) \tag{36.4}$$

$$= \partial_p u_\alpha. \tag{36.5}$$

In the last line we defined a normalized tangent vector to the world-line, u^α. By the definition of $\mathcal{L}$,

$$u^\alpha u_\alpha = -1. \tag{36.6}$$

Not surprisingly, u^α is the particle 4-velocity and the equations of motion simply assert that the 4-velocity (and hence 4-momentum) of the particle is constant along its world-line. Using the time t to parameterize the world line of a particle with constant 3-vector velocity $\mathbf{v}$,

$$r^0(t) = ct, \qquad \mathbf{r}(t) = \mathbf{v}t, \tag{36.7}$$

$$\partial_t r^0(t) = c, \qquad \partial_t \mathbf{r}(t) = \mathbf{v}, \tag{36.8}$$

$$\mathcal{L} = \sqrt{c^2 - \mathbf{v}\cdot\mathbf{v}} = \frac{c}{\gamma} \tag{36.9}$$

we arrive at the familiar way of expressing the 4-velocity:

$$u^0 = \gamma, \qquad \mathbf{u} = \gamma\mathbf{v}/c. \tag{36.10}$$

The equations of motion for the relativistic string are found analogously, by applying Hamilton's principle to the action

$$S[s] = C\int \mathcal{L}\, dp\, dq \tag{36.11}$$

$$\mathcal{L} = \sqrt{-a^{\alpha\beta}a_{\alpha\beta}} \qquad a^{\alpha\beta} = \frac{\epsilon^{ab}}{\sqrt{2}}\,\partial_a s^\alpha\,\partial_b s^\beta. \tag{36.12}$$

This is an instance of an action given as the integral over two variables, something we first encountered in lecture 13 (the non-relativistic elastic string). The Euler-Lagrange equations now have two derivative terms,

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial s^\alpha} - \partial_p\left(\frac{\partial\mathcal{L}}{\partial(\partial_p s^\alpha)}\right) - \partial_q\left(\frac{\partial\mathcal{L}}{\partial(\partial_q s^\alpha)}\right), \qquad \alpha = 0,1,2,3 && (36.13)\\ &= \frac{\partial\mathcal{L}}{\partial s^\alpha} - \partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a s^\alpha)}\right), && (36.14)\end{aligned}$$

where the last line abbreviates the sum over both terms using the Einstein summation convention (for repeated latin indices). Computing the derivatives (assigned as homework) one obtains the equations

$$0 = \epsilon^{ab}(\partial_a v^{\alpha\beta})(\partial_b s_\beta), \tag{36.15}$$

where

$$v^{\alpha\beta} = a^{\alpha\beta}/\mathcal{L} \tag{36.16}$$

is a rescaling of the surface element, analogous to the rescaling of the line element for particles that defined the 4-velocity u^α. But the analogy ends there, because it is not this $v^{\alpha\beta}$ that has zero rate-of-change over the world-surface, but the more complex combination (36.15).

36.1.2 Relativistic strings with high symmetry

In lecture 39 will see that by taking advantage of the freedom we have in choosing parameters, it is possible to write down the most general solution to the non-linear equations of motion (36.15). In the meantime, we will construct some solutions that have special symmetries.

A straight, infinite string in uniform motion is a solution because we have the parameterization from lecture 35

$$s^\alpha(x,t) = (ct\,, x\,, vt\,, 0), \tag{36.17}$$

for which $a^{\alpha\beta}$ is constant and therefore also $\mathcal{L}$ and $v^{\alpha\beta}$. The equations of motion (36.15) are solved because both parameter derivatives of $v^{\alpha\beta}$ vanish everywhere on the surface.

A more interesting symmetric case is a string moving only in the (x,y) plane and invariant with respect to continuous rotations about the origin at all times — a circle. The only property of this string that is not determined by symmetry is the radius $r(t)$ of the circle at different times t — this "time" being the coordinate time of the observer at rest with respect to the (x,y) plane that holds the string. We will use t

as one parameter of the world-surface and the angle θ, used by the observer to distinguish the points of the circle, as the other parameter:

$$s^\alpha(\theta, t) = (ct\,, r(t)\cos\theta\,, r(t)\sin\theta\,, 0). \tag{36.18}$$

Substituting this into the equations of motion (36.15) would give us a differential equation for $r(t)$ that we would then have to solve to complete the solution. We will use a slightly more direct approach, where we first simplify the action S for this symmetric class of strings.

Here are the two tangent vectors to the world-surface:

$$\partial_\theta s^\alpha = (0\,, -r\sin\theta\,, r\cos\theta\,, 0) \tag{36.19}$$

$$\partial_t s^\alpha = (c\,, \dot r\cos\theta\,, \dot r\sin\theta\,, 0). \tag{36.20}$$

From these we compute the square of the Lagrangian,

$$\begin{aligned}
\mathcal{L}^2 &= -a^{\alpha\beta}a_{\alpha\beta} && (36.21)\\
&= (\partial_\theta s^\alpha\,\partial_t s_\alpha)^2 - (\partial_\theta s^\alpha\,\partial_\theta s_\alpha)(\partial_t s^\beta\,\partial_t s_\beta) && (36.22)\\
&= (0)^2 - (r^2)(-c^2 + \dot r^2) && (36.23)\\
&= r^2(c^2 - \dot r^2), && (36.24)
\end{aligned}$$

and the action:

$$\begin{aligned}
S[r] &= C\int_0^{2\pi} d\theta \int_{t_1}^{t_2} r\sqrt{c^2 - \dot r^2}\,dt && (36.25)\\
&= 2\pi C\int_{t_1}^{t_2} r\sqrt{c^2 - \dot r^2}\,dt && (36.26)\\
&= 2\pi C\int_{t_1}^{t_2} \mathcal{L}_1(r, \dot r)\,dt. && (36.27)
\end{aligned}$$

Because of the symmetry restriction, the action is a functional of a single function of just one argument, $r(t)$, as we have in the mechanics of a single degree of freedom. The Euler-Lagrange equation for this single degree of freedom,

$$0 = \frac{\partial\mathcal{L}_1}{\partial r} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}_1}{\partial\dot r}\right), \tag{36.28}$$

leads (after some simplification) to the following non-linear, second-order differential equation:

$$r\ddot r - \dot r^2 + c^2 = 0. \tag{36.29}$$

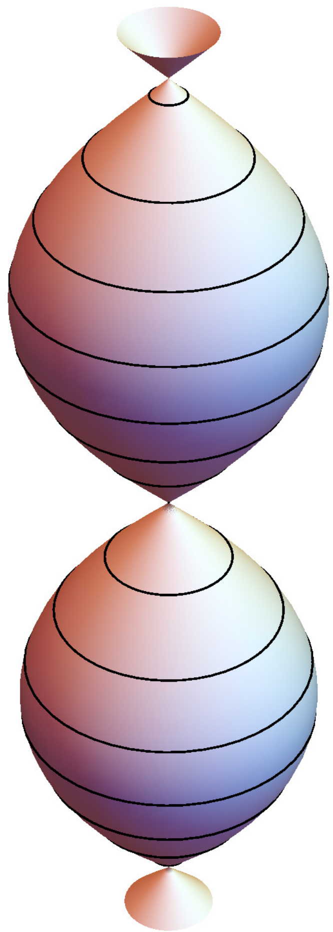
The most general solution of this equation is

$$r(t) = R\sin\left(\frac{t - t_0}{R/c}\right). \tag{36.30}$$

No doubt you will have seen many expressions involving sinusoidal functions in the course of your physics education, but this is probably the first time where the amplitude and period of the oscillation are related by a factor of c, the speed of light! The transverse (radial) velocity of the string

$$\dot r(t) = c\cos\left(\frac{t - t_0}{R/c}\right), \tag{36.31}$$

has magnitude c at the times when its radius is instantaneously zero (half-way between times of maximum radius). In lecture 37 we will learn how to compute the conserved energy of relativistic strings. When evaluated for the oscillating circular string, the energy is proportional to R — the maximum radius — and consistent with there being a fixed linear energy density in the string. All this energy is purely kinetic in nature at the other extreme, when the elements of the string have shrunken to zero length and are moving with the speed of light.



World-surface of a circular relativistic string. The time axis is vertical and the black circles are snapshots of the string at particular instants of time. At the three light-cone-like events shown, the transverse velocity of the string instantaneously attains the velocity of light.