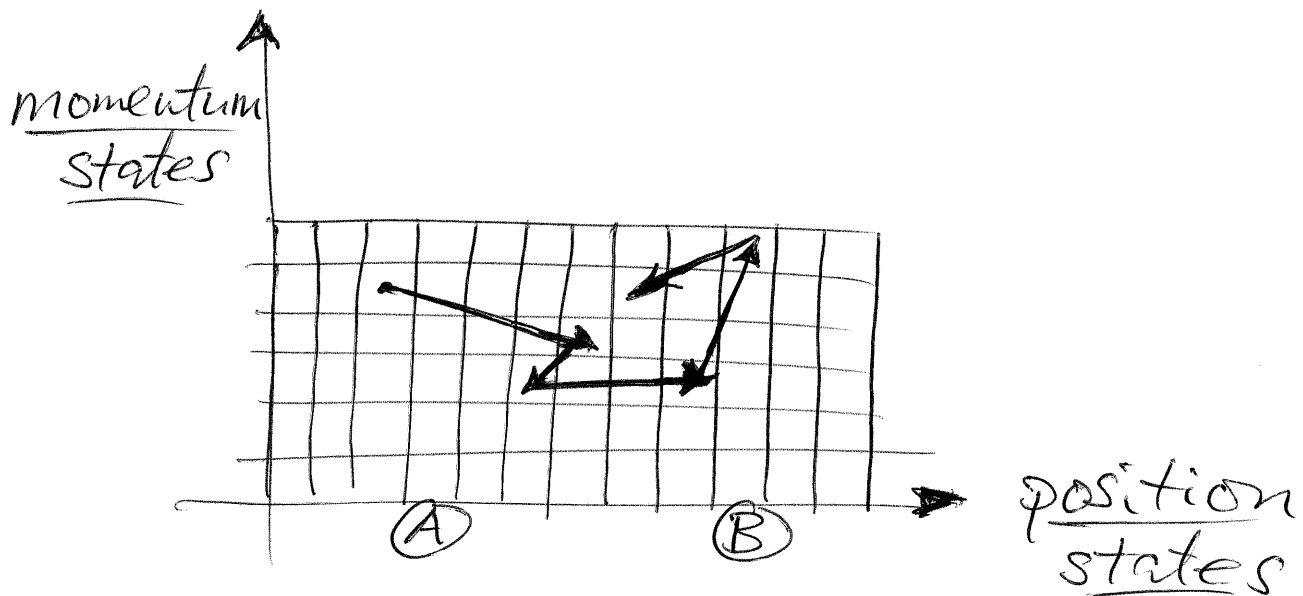


Lecture 35

In the previous lecture we explained the curious statistics of the position of a particle in a 2D potential like this:

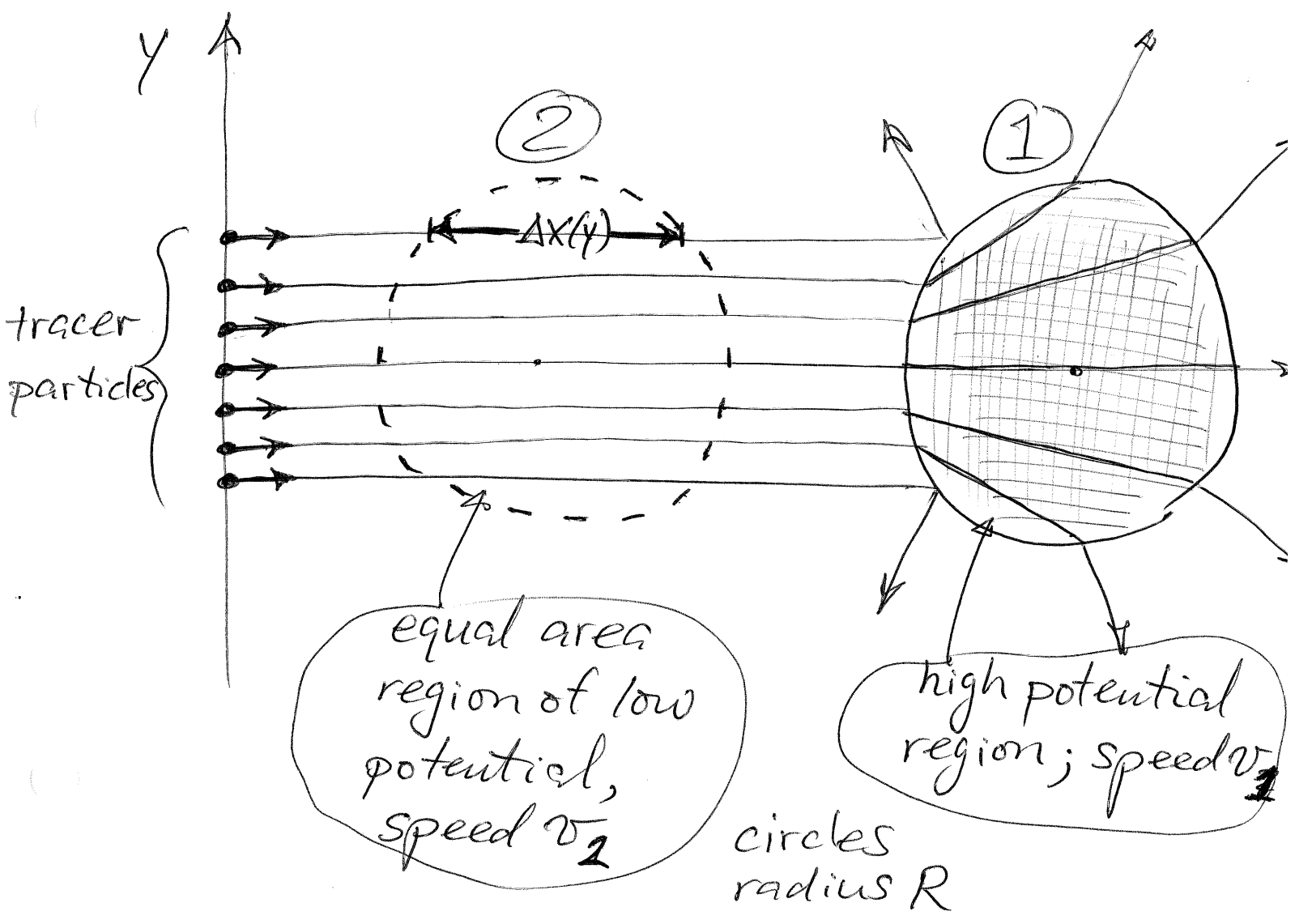


Each cell represents one micro-state, and the system hops from one microstate to another over the course of time. Our central hypothesis states that all cells are visited equally. Moreover, we found that over each position state, say (A)

or (B), there are the same number of momentum states (for the special case of motion in 2D). Our hypothesis then implies that states with positions (A) and (B) ~~will~~ will be equally likely over the course of time — even when the corresponding potentials are different and the particle speeds are different.

We would like to make this fact at least plausible while working purely within mechanics. Start by making two reasonable assumptions:

- the distribution of velocities is isotropic
- for any given velocity, trajectories starting with that velocity have a uniform distribution of "impact parameter" (y-axis) :



fact we hoped to prove.



Now that we have made the hypothesis, of all microstates being equally likely, more plausible, let's apply the principle to an "ideal gas" of N point particles in 3D. By stating the gas to be "ideal" we mean that it is sufficiently dilute so that the energy of a particle is essentially independent of its position. [We cannot insist there are no intermolecular forces whatsoever,

since it is the collisions ^(35.8) between particles that randomizes the momenta ~~over~~ ~~the~~ in accordance with our microstate hypothesis.]

The number of microstates will be a function of two important parameters:

- volume of the box, V
- total energy of gas, E

$$\Omega(V, E) = (\text{num. microstates})$$

$$= \left(\begin{array}{l} \text{num. pos.} \\ \text{states of} \\ \text{particle 1} \end{array} \right) \times \left(\begin{array}{l} \text{num. pos.} \\ \text{states of} \\ \text{particle 2} \end{array} \right) \dots \left(\begin{array}{l} \text{num. pos.} \\ \text{states of} \\ \text{particle N} \end{array} \right)$$

\times (num. of momentum (35.9)
states of N particles
with total energy between
 E and $E + \delta E$)

Each of the first N factors

is

$$\left(\frac{V}{\delta x \delta y \delta z} \right) .$$

Write the momentum states
factor as the difference

$$(\dots) = W(E + \delta E) - W(E)$$

where

$$W(E) = \left(\begin{array}{l} \text{num. of momentum} \\ \text{states of } N \text{ particles} \\ \text{with energy } \underline{\text{less than}} \text{ } E \end{array} \right)$$

In an ideal gas of (35.10)
 N point particles (in 3D) the
energy is simply

$$E = \frac{1}{2m} (P_{1x}^2 + P_{1y}^2 + P_{1z}^2 + \dots + P_{Nx}^2 + P_{Ny}^2 + P_{Nz}^2)$$

For simplicity we're considering
a gas where all the particles
have the same mass m . The
equation above describes the
surface of a "sphere" in $3N$
dimensions. This sphere exists
in momentum space in which
we have a fine grid for the

(35.11)

purpose of counting microstates. The number of grid points inside the hyper-sphere equals the hyper-volume of the sphere divided by the hyper-volume per grid point—a constant factor we don't care about. Since the momentum radius of the hyper-sphere is $R = \sqrt{2mE}$, and the hyper volume of the sphere in $3N$ dimensions is (constant) R^{3N} , we obtain

$$W(E) \propto R^{3N} \propto E^{\frac{3N}{2}},$$

and for the difference,

(35.12)

$$W(E + \delta E) - W(E) \propto E^{\frac{3N}{2} - 1} \delta E$$

This is, up to a multiplicative constant, the number of momentum microstates of the gas.

Multiplying by the position states we arrive at our goal:

$$\Omega(N, E) \propto V^N E^{\frac{3N}{2} - 1}$$