Lecture 35

The analysis of the pure magneto-statics scenario, in a moving frame, is almost the same as the electro-static case we analyzed in the previous lecture. The roles of \( \vec{E} \) and \( \vec{B} \) are interchanged, there are some sign differences, and c's to keep the units correct:

\[
\begin{align*}
\vec{B} &= -B \hat{z} \\
\vec{E} &= 0 \\
\vec{V} &= u \hat{x} \\
\vec{V} \times \vec{B} &= u B \hat{y}
\end{align*}
\]

\[
\begin{align*}
B_x &= B_x = 0 \\
B_y' &= 0 \\
B_z' &= y B_z = -y B \\
E_x' &= E_x = 0 \\
E_y' &= y B \hat{y} \\
E_z' &= 0
\end{align*}
\]

\[ \Rightarrow \vec{E}' = y u B \hat{y} \quad \vec{B}' = -y B \hat{z} \]
This time the circulation of $\mathbf{E}'$ around $C'$ is not zero (as it is in electrostatics):

$$\oint_{C'} \mathbf{E}' \cdot d\mathbf{r}' = + yuvBL$$

The time-dependent entity is
now the flux of magnetic field:

\[ \Delta \Phi_B' = \frac{-\gamma B \cdot L \Delta t'}{\Delta t'} = -\gamma B V L \]

This is (-1) times the discrepancy in the circulation of \( \vec{E}' \), which suggest the following amendment:
\[
\oint_{C'} \vec{E}' \cdot d\vec{r}' + \frac{d}{dt'} \left( \int_{S'} \vec{B}' \cdot d\vec{a}' \right) = 0
\]

After bringing the time derivative inside the surface integral and using Stokes' law (exactly as earlier) we obtain:

\[
\int_{S'} \left( \nabla \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} \right) \cdot d\vec{a}' = 0
\]

The electrostatics law \( \nabla \times \vec{E} \) should therefore be amended as

\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0
\]
We used particularly simple and symmetric field configurations to arrive at the two amendments of the laws for the circulation of \( \vec{E} \) and \( \vec{B} \). Before we accept them as general laws we should subject them to the ultimate theoretical test: invariance with respect to Lorentz transformations (i.e. the same laws apply in any inertial frame). Recently we saw how \( \vec{E} \) and \( \vec{B} \) transform; what remains is to work out how the space and time derivatives in these laws transform.
Warm-up exercise: Suppose we have a function of two variables $x$ and $y$, say $f(x, y)$. We already know what the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ mean. But suppose we are interested in describing the $x$-$y$ plane with another pair of variables $x', y'$. As a concrete example, let $x', y'$ be coordinates in a rotated frame:

\[ x = \cos \theta x' - \sin \theta y' \]
\[ y = \sin \theta x' + \cos \theta y' \]
So we can think of \( f \) as actually a function of \( x' \) and \( y' \), since \( x \) and \( y \) are functions of these variables:

\[
\frac{df}{dx'} = \frac{df(x(x', y'), y(x', y'))}{dx'}
\]

\[
= \frac{df}{dx} \frac{dx}{dx'} + \frac{df}{dy} \frac{dy}{dx'}
\]

\[
= \frac{df}{dx} \cos \Theta + \frac{df}{dy} \sin \Theta
\]

Similarly:

\[
\frac{df}{dy'} = \frac{df}{dx} (-\sin \Theta) + \frac{df}{dy} \cos \Theta
\]
Since \( \sin \theta \) and \( \cos \theta \) are constants (the angle of rotation is fixed) we can rearrange things like this:

\[
\frac{\partial}{\partial x} f = (\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}) f
\]

\[
\frac{\partial}{\partial y} f = (-\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}) f
\]

Finally, \( f \) is an arbitrary function so what we've found is a transformation rule for partial derivatives.