Lecture 35: May 1

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35.1 Geometric action principles and relativity (continued)

35.1.1 Relativistic string action

Before we derive equations of motion and conserved quantities for the particle action (lecture 34), we extend our geometric construction to the string, where the world-line is replaced by a world-surface. As the calculation of surface area elements may not be as familiar as lengths of line elements, we recall how this is done for area elements in ordinary three dimensional Euclidean space.

Consider a surface (in three dimensions) $\mathbf{s}(p, q)$ parameterized by p and q, and an element of surface defined by the two (not necessarily orthogonal) tangent vectors

$$\mathbf{u}(1) = \partial_p \mathbf{s}\, dp, \qquad \mathbf{u}(2) = \partial_q \mathbf{s}\, dq. \tag{35.1}$$

These form a parallelogram and using the cross product we can compute their squared area:

$$\begin{aligned} (A\, dp\, dq)^2 &= (\mathbf{u}(1) \times \mathbf{u}(2)) \cdot (\mathbf{u}(1) \times \mathbf{u}(2)) && (35.2)\\ &= (\mathbf{u}(1) \cdot \mathbf{u}(1))\, (\mathbf{u}(2) \cdot \mathbf{u}(2)) - (\mathbf{u}(1) \cdot \mathbf{u}(2))\, (\mathbf{u}(2) \cdot \mathbf{u}(1))\, . && (35.3) \end{aligned}$$

For our surface $s(p, q)$ in Minkowski space the tangent vectors are

$$u(1) = \partial_p s\, dp, \qquad u(2) = \partial_q s\, dq, \tag{35.4}$$

and the area of the element is

$$(A\, dp\, dq)^2 = u^\alpha(1) u_\alpha(1)\, u^\beta(2) u_\beta(2) - u^\alpha(1) u_\alpha(2)\, u^\beta(2) u_\beta(1). \tag{35.5}$$

It will turn out to be very useful, as in the case of the proper time, to write this as the square of something. The object that serves this purpose is the following tensor:

$$a^{\alpha\beta}\, dp\, dq = \frac{1}{\sqrt{2}} \left(u^\alpha(1) u^\beta(2) - u^\alpha(2) u^\beta(1) \right). \tag{35.6}$$

It's easy to check that

$$(A\, dp\, dq)^2 = \left(a^{\alpha\beta}\, dp\, dq \right) \left(a_{\alpha\beta}\, dp\, dq \right), \tag{35.7}$$

and therefore

$$A\, dp\, dq = \sqrt{a^{\alpha\beta} a_{\alpha\beta}}\; dp\, dq. \tag{35.8}$$

The second-order tensor $a^{\alpha\beta}$ is antisymmetric in its two indices. Comparing (35.4) with (35.6) we obtain

$$a^{\alpha\beta} = \frac{1}{\sqrt{2}}\left(\partial_p s^\alpha \, \partial_q s^\beta - \partial_q s^\alpha \, \partial_p s^\beta\right). \tag{35.9}$$

Writing this in terms of the Levi-Civita symbol ϵ^{ab}, ($\epsilon^{pq} = 1$, $\epsilon^{qp} = -1$, $\epsilon^{pp} = \epsilon^{qq} = 0$) and the Einstein summation convention,

$$a^{\alpha\beta} = \frac{\epsilon^{ab}}{\sqrt{2}} \, \partial_a s^\alpha \, \partial_b s^\beta. \tag{35.10}$$

we see how this would generalize to world "surfaces" of any dimension.

By analogy with the world line, we take the integral of the area elements as the definition of the world surface's action:

$$S[s] = C \int \mathcal{L} \, dp \, dq \tag{35.11}$$

$$\mathcal{L} = \sqrt{a^{\alpha\beta} a_{\alpha\beta}} \qquad a^{\alpha\beta} = \frac{\epsilon^{ab}}{\sqrt{2}} \, \partial_a s^\alpha \, \partial_b s^\beta. \tag{35.12}$$

The C in this definition sets the scale of the action, analogous to the constant Mc in the particle action.

35.1.2 Signs and tachyons

Now is a good time to take account of the $\pm$ signs we have in the geometry of space-time, signs that are absent in Euclidean space. Our definition of the particle action, as the integral of the proper time along the world-line,

$$S[r] = Mc \int d\tau = Mc \int \sqrt{-(\partial_p r^\alpha)(\partial_p r_\alpha)} \, dp, \tag{35.13}$$

is well defined as long as the tangent vector to the world-line, $\partial_p r^\alpha$, is everywhere time-like. Hypothetical particles that violate this restriction are called "tachyons".

We must check if any analogous restrictions apply to the world-sheet of a relativistic string. Consider an observer at time t. This observer sees the world-sheet at that particular instant as a string. Also suppose this observer is mostly interested in a short piece of the string, so short that it may be approximated as straight. What motion does this piece of string display?

Because the string has no features along it length, the only motion that *can* be observed is transverse motion. Suppose the axis defined by the short piece of string in the observer's frame is x, and the direction of transverse motion is along the y-axis. Given the instantaneous transverse velocity v measured by the observer, we can construct an approximate parameterization of the string valid for the short piece of string over a short period of time. The position x along the string and the observer's time t will serve as the parameters p and q. A small "patch" of world-surface, written in terms of these parameters and having the required motion, will have the following coordinates:

$$s^0(x,t) = ct \tag{35.14}$$
$$s^1(x,t) = x \tag{35.15}$$
$$s^2(x,t) = vt \tag{35.16}$$
$$s^3(x,t) = 0. \tag{35.17}$$

Here are the corresponding tangent vectors (written as 4-vectors),

$$\partial_x s^\alpha = (0,1,0,0) \tag{35.18}$$
$$\partial_t s^\alpha = (c,0,v,0), \tag{35.19}$$

and the computation of the square of the area:

$$a^{\alpha\beta}a_{\alpha\beta} = (\partial_x s^\alpha\, \partial_x s_\alpha)(\partial_t s^\beta\, \partial_t s_\beta) - (\partial_x s^\alpha\, \partial_t s_\alpha)^2 \tag{35.20}$$
$$= (+1)(-c^2 + v^2) - (0)^2 \tag{35.21}$$
$$= -c^2 + v^2. \tag{35.22}$$

Because this is negative for a string with sub-light-speed transverse velocity, we should modify our definition of the string action:

$$S[s] = C \int \sqrt{-a^{\alpha\beta}a_{\alpha\beta}}\, dp\, dq. \tag{35.23}$$

And since $a^{\alpha\beta}a_{\alpha\beta}$ is Lorentz-invariant, this definition gives a sensible non-negative argument for the square root for all observers. The question of the sign of the argument of the square root will come up again, when we recall how Hamilton's principle and classical motion arises from quantum amplitudes.

Question: What should be the units of the constant C so that S has units of action?