

## Lecture 34: April 28

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## 34.1 Geometric action principles and relativity

It will not have escaped your notice that time plays a special role in all the formalisms of mechanics we have studied in this course. In the Lagrangian formalism we consider systems specified by completely arbitrary sets of generalized coordinates (positions, angles, etc.) and assert that the Lagrangian is a function of these coordinates in addition to their time derivatives. The distinguished role of time carries over to the Hamiltonian formalism, where half of all the variables — the conjugate momenta — are defined in terms of time derivatives. And conservation laws, by their nature, are statements specifically about constancy in time.

It is no accident, of course, that all the pre-Einstein formalisms of mechanics gave a special status to time. And as long as we apply these formalisms to systems where velocities are small on the scale of the speed of light, there is no need to replace them. In fact, idealizations such as “a rigid body” break down already when velocities (of rotation about the center of mass) exceed the speed of *sound*. On the other hand, mechanics is valuable not just for the technical and analytic tools it provides (in the low-speed realms where it applies), but for its insights on the foundations of physical theories. Such was the case with Hamilton’s principle, whose explication required quantum mechanics.

One response to Einstein’s “demotion” of time would be to study how the existing formalisms of mechanics can be revised to be consistent with special relativity. Instead of this, and keeping with the foundational perspective, we will take Einstein’s geometrical world-view as inspiration for the design of physical theories. We start by giving the dynamics of a point particle a purely geometrical interpretation. Preserving the principles while generalizing the geometry, we then propose a very interesting mechanical system: a fully relativistic “string”. The quantum analogs of such objects have been proposed as the fundamental building blocks of matter.

### 34.1.1 Notation

A particle in motion describes a curve in space-time. We will represent such “world-lines” as events  $r(p)$  in space-time parameterized by a real number  $p$ . The action for the particle should not depend on how the world-line — a geometrical object — is parameterized. We will check that our action is unchanged when re-parameterized by a different parameter  $p' = f(p)$ , where  $f$  is an arbitrary monotonic function.

We use greek index superscripts  $\alpha, \beta, \dots$  for the space-time components of 4-vectors, such as the location  $r^\alpha$  of an event. The first component is the time-component, so  $r^0$  would be the time of an event (times  $c$ , the speed of light), and components  $(r^1, r^2, r^3)$  the position in space — what we have been denoting by  $\mathbf{r}$  in previous lectures. A standard way to express the Minkowski inner product rule is with superscript *and*

subscript indices. For example,  $r_\alpha$  is defined by

$$r_\alpha = g_{\alpha\beta} r^\beta, \quad (34.1)$$

where we have used the Einstein summation convention for repeated indices, and  $g$  is the Minkowski metric tensor with diagonal components  $g_{00} = -1$ ,  $g_{ii} = +1$ ,  $i = 1, 2, 3$ . The Minkowski inner product always involves subscript/superscript pairs:

$$r^\alpha r_\alpha = r^0 r_0 + r^1 r_1 + r^2 r_2 + r^3 r_3 = -(r^0)^2 + (r^1)^2 + (r^2)^2 + (r^3)^2. \quad (34.2)$$

Scalar quantities have no un-summed indices and are invariant with respect to Lorentz transformation. The scalar above is  $-c^2$  times the square of the proper time between the origin and event  $r$ .

Increasing by one the dimension of a world-line produces a world-surface. To specify these we need two real parameters,  $p$  and  $q$ . Events on the world-surface are written  $s(p, q)$ . The action we construct for world-surfaces will also be purely geometric and should be unchanged when we transform to another pair of parameters,  $(p', q') = (f_1(p, q), f_2(p, q))$ . An observer at time  $t$  will see the world-surface events where  $s^0(p, q) = ct$ . This implicitly defines  $q$  as a function of  $p$ , and so the world-surface at a specific time is a curve  $\mathbf{s}(p, q(p))$  in space parameterized by one parameter. These objects are called “strings”.

At some point we will construct densities in space-time corresponding to the energy and momentum of particles or strings. Our notation for location in space-time, the argument of these densities, is the four-vector  $x^\alpha$ , and should not be confused with the particular events occupied by the particle ( $r^\alpha$ ) or string ( $s^\alpha$ ). We will be taking partial derivatives  $\partial/\partial x_\alpha$  of these densities, and also partial derivatives  $\partial/\partial p$  and  $\partial/\partial q$  with respect to the parameters. The former type of derivative will be abbreviated  $\partial_\alpha$  and the latter  $\partial_a$ , where latin indices are reserved for  $p$  and  $q$ . An example of a density that combines these types of derivatives is

$$\rho(x) = \partial_p \delta^4(x - r(p)) = -(\partial_p r^\alpha) \partial_\alpha \delta^4(x - r(p)), \quad (34.3)$$

where  $\delta^4(x - r)$  is a Dirac density in space-time (unit weight located at  $x = r$ ) and the summation on  $\alpha$  comes about from the application of the multi-variable chain rule to the  $\partial_p$  derivative.

### 34.1.2 Relativistic particle action

The simplest Lorentz-invariant scalar one can define for a world line is its length, that is, the net proper time elapsed for an observer with that world line. For our parameterized world line  $r(p)$ , the proper time  $d\tau$  elapsed on an infinitesimal element of extent  $dp$  in the parameter is the following Minkowski inner product:

$$c^2 d\tau^2 = -(\partial_p r^\alpha dp)(\partial_p r_\alpha dp). \quad (34.4)$$

The integral of  $d\tau$  for world lines  $r(p)$  that join two fixed events is our geometrical candidate for an action functional  $S[r]$ :

$$S[r] = Mc \int \mathcal{L} dp, \quad (34.5)$$

$$\mathcal{L} = \sqrt{-(\partial_p r^\alpha)(\partial_p r_\alpha)}. \quad (34.6)$$

As a mathematical expression we have tried to preserve the form we are familiar with: an integral of a Lagrangian function that depends on at most first derivatives of the variables. The integration variable is not time as in ordinary mechanics — that would make the definition explicitly non-Lorentz invariant — but the arbitrary world-line parameter  $p$ . As a geometrical entity  $S$  should be independent of how we choose to parameterize the world line, but this is also easily checked. When re-expressed in terms of the new parameter  $p' = f(p)$ , the derivatives transform as  $\partial_p = (df/dp) \partial_{p'}$  while in the integral  $dp = (df/dp)^{-1} dp'$ . Finally, the momentum factor  $Mc$ , when multiplied by  $c$  times the elapsed time, gives  $S$  the correct units.