Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 33: April 26

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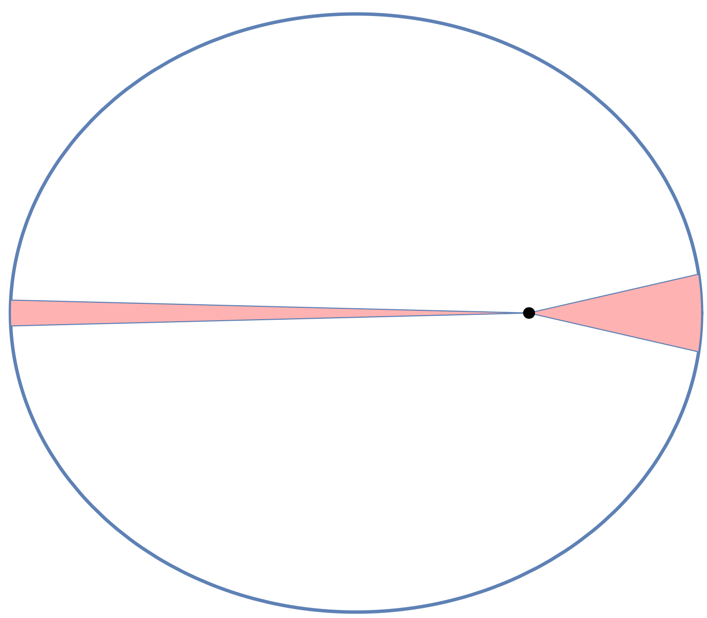
Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

33.1 Gravitational two-body orbits (continued)

33.1.1 Kepler's laws

Kepler's first law states that the "position vector sweeps out equal areas in equal time". This refers to the *relative position* vector, what we have been denoting $\mathbf{r}$ or by polar coordinates (r, θ) in the orbital plane.



Areas swept out by the orbit over equal time periods Δt at the two extremes of the orbital speed.

Kepler's first law is a direct consequence of the constancy of L_z. Computing the area swept out in a short time Δt,

$$\Delta A = \left(\frac{\Delta\theta}{2\pi}\right)\pi r^2 = \frac{1}{2} r^2 \dot{\theta}\, \Delta t, \tag{33.1}$$

we find, using the definition of L_z (lecture 29),

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_z}{2\mu} = \text{constant.} \tag{33.2}$$

Applying the equal-areas-in-equal-times law to the computation of the entire ellipse area of the orbit, we obtain

$$A = \int_0^T \dot{A}\, dt = \frac{L_z}{2\mu} T, \tag{33.3}$$

where T is the period of the orbit. By expressing the area in terms of the major/minor axes, $A = \pi ab$, this produces the formula (assigned as homework)

$$G(M_1 + M_2)T^2 = 4\pi^2 a^3, \tag{33.4}$$

known as Kepler's second law. You will recognize this as exactly the law we derived for circular orbits (lecture 30), but with the semi-major axis a replacing the circular radius r_0.

33.1.2 Orbits in the center of mass frame

Let's describe the motion of the two bodies in the center-of-mass frame. Since $\dot{\mathbf{R}} = 0$, we choose to place the center of mass at the origin, $\mathbf{R} = 0$. In this coordinate system (lecture 29),

$$\mathbf{r}_1 = +\left(\frac{M_2}{M_1 + M_2}\right)\mathbf{r} \tag{33.5}$$

$$\mathbf{r}_2 = -\left(\frac{M_1}{M_1 + M_2}\right)\mathbf{r}. \tag{33.6}$$

The two bodies have positions parallel to the relative position vector, with mass-ratio dependent scale factors. At any time, the more massive body will be closer to the origin (center of mass) than the lighter body.

As a not atypical example, let's construct the trajectories of a binary star system with equal mass stars, where

$$\mathbf{r}_1 = \frac{1}{2}\mathbf{r} = -\mathbf{r}_2. \tag{33.7}$$

For concreteness we'll take $\epsilon = \sqrt{3}/2$ so that

$$\frac{a}{b} = \frac{1}{\sqrt{1-\epsilon^2}} = 2. \tag{33.8}$$

We start by drawing the orbit of $\mathbf{r}$, an ellipse of aspect ratio 2. The origin will divide the major axis into the ratio (lecture 32)

$$\frac{r_{\max}}{r_{\min}} = \frac{1+\epsilon}{1-\epsilon} \approx 14. \tag{33.9}$$

With this information we can construct corresponding pairs of positions for $\mathbf{r}_1$ and $\mathbf{r}_2$ from the ellipse of the relative position $\mathbf{r}$. The result, a pair of intersecting ellipses, is what a time-lapse photograph of the binary star system would show when viewed perpendicular to the orbital plane.

Drawing exercise: For the parameters above, construct corresponding pairs of star positions from the relative-separation ellipse.

33.1.3 Gravitational scattering and "gravity assist"

In non-gravitational contexts the term "scattering" is often applied to two-body motion when the initial and final states are asymptotically free — along straight lines. This is also an apt term for a particular mode of spacecraft maneuvering called "gravity assist".

As we observed in the discussion of the unbound, hyperbolic orbits (lecture 32), the linear asymptotes do not pass through the origin, but a point $(x_0, 0)$ offset from the origin. This is explained by the fact that the angular momentum L_z is defined with respect to the origin (for the relative separation $\mathbf{r}$) and would always evaluate to zero if the asymptotes passed through the origin. In order to have a nonzero L_z the asymptotes must be shifted relative to lines through the origin by an amount b called the *impact parameter*. We will shortly see that this definition of b agrees algebraically with the hyperbola parameter b (lecture 32).

We can relate the impact parameter b to the other orbital parameters using the asymptotic limits of the conserved angular momentum and energy. Both of these can be expressed in terms of the asymptotic speed v_∞ (of relative motion):

$$L_z = (\mathbf{r} \times \mathbf{p})_z = b\mu v_\infty \tag{33.10}$$

$$E = \frac{1}{2}\mu v_\infty^2. \tag{33.11}$$

Eliminating v_∞ between these equations,

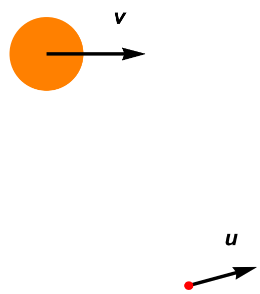
$$\frac{L_z^2}{E} = 2\mu b^2, \tag{33.12}$$

and using formulas for r_0 and E from lecture 32, we obtain

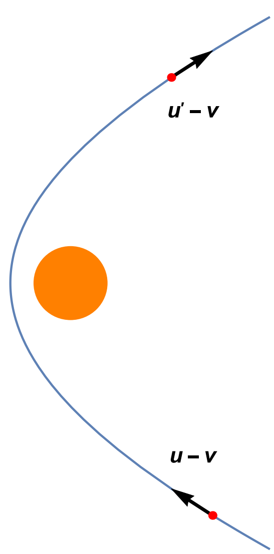
$$b^2 = \frac{L_z^2}{2\mu E} = \frac{A\mu r_0}{(\epsilon^2 - 1)A\mu/r_0} = \frac{r_0^2}{\epsilon^2 - 1}. \tag{33.13}$$

This is the same as the b parameter that appeared in the formula for the hyperbola.

In the gravity assist maneuver, a spacecraft is in a near collision course with a planet (or moon) whose mass is of course much greater. In the rest frame of the solar system, the two bodies have nearly parallel velocities, the planet being slightly faster and playing catch-up.



By analyzing gravity assist as a scattering process very few details are relevant. We can see this when transforming to the center-of-mass frame of the planet-spacecraft system, which is practically indistinguishable from the planet rest frame. The orbit of the spacecraft will be a hyperbola with the planet at the origin and the asymptotes of the spacecraft trajectory intersecting at a point *behind* the planet (behind with respect to its motion in the solar system).



In the center-of-mass frame the spacecraft's initial asymptotic velocity $\mathbf{u}-\mathbf{v} = (u_x - v)\hat{\mathbf{x}} + u_y\hat{\mathbf{y}}$ is converted to the final asymptotic velocity $\mathbf{u}'-\mathbf{v} = (-u_x + v)\hat{\mathbf{x}} + u_y\hat{\mathbf{y}}$ as it swings around the trailing side of the planet. This is purely the result of symmetry and the planet's much larger mass, and independent of the details of the force law. On the other hand, the relationship between r_{min} and b does depend on the force law and is something you now know how to calculate in the case of gravity.

Transforming the spacecraft's final asymptotic velocity back to the solar system frame we obtain $\mathbf{u}' = (-u_x + 2v)\hat{\mathbf{x}} + u_y\hat{\mathbf{y}}$. Comparing final and initial spacecraft speeds, we obtain

$$\left(\frac{u'}{u}\right)^2 = \frac{(2v-u_x)^2 + u_y^2}{u^2} \tag{33.14}$$

$$= \left(\frac{2v}{u}\right)^2 - 4\left(\frac{v}{u}\right)\left(\frac{u_x}{u}\right) + 1 \tag{33.15}$$

$$= \left(\frac{2v}{u} - 1\right)^2 + 4\left(\frac{v}{u}\right)(1-\cos\theta)\,, \tag{33.16}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ in the solar system frame. This shows that $u' > u$ whenever $v > u$.

Question: How can the spacecraft speed increase and yet have energy conserved?

A succession of gravity assist scatterings was used to propel the Voyager spacecraft to the outer planets and beyond. Voyager was able to "climb out" of the sun's deep gravitational "well" by falling into the gravitational "wakes" of moving planets and getting a brief but powerful pull in the direction of the planet's motion. One can make a rough analysis of such schemes by combining the two-body spacecraft/planet behavior on short time/length scales (hyperbolic orbits) with the two-body spacecraft/sun (and planet/sun) elliptical orbits on long time/length scales.