Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 32: April 24

Instructor: Veit Elser

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

32.1 Gravitational two-body orbits

Without loss of generality (by choice of coordinates) we set $\theta_0 = 0$ in the orbit formula (lecture 31):

$$1/u(\theta) = r(\theta) = \frac{r_0}{1 + \epsilon\cos\theta}, \qquad r_0 = \frac{L_z^2}{A\mu}, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}, \quad A = GM_1M_2. \tag{32.1}$$

The eccentricity ϵ is a dimensionless parameter that, in addition to being strongly linked to the shape of the orbit, is related to the conserved energy E. The term is apt because $\epsilon = 0$ corresponds to a circular orbit. Notice that flipping the sign of ϵ is equivalent to redefining the angle by $\theta \to \theta + \pi$. We may therefore assume $\epsilon \geq 0$.

32.1.1 Bound and unbound orbits

Using the orbit equation and the definitions of the parameters, the expression for the energy derived in lecture 31 can be written compactly as follows (assigned as homework):

$$E = \frac{L_z^2}{2\mu}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - Au \tag{32.2}$$

$$= \frac{A}{2r_0}(\epsilon^2 - 1). \tag{32.3}$$

We see that if $0 \leq \epsilon < 1$ the energy E is negative, and therefore inconsistent with the scenario where the distance between the bodies grows without bound (in the limit it would be purely kinetic and therefore positive). The case $0 \leq \epsilon < 1$ therefore corresponds to bound orbits.

32.1.2 Bound orbits: the ellipse

A simple calculation shows that the shape of bound orbits ($0 \leq \epsilon < 1$) is always an ellipse. From the orbit equation we see that the distance ranges between two extremes:

$$r_{\min} = r(0) = \frac{r_0}{1+\epsilon}, \qquad r_{\max} = r(\pi) = \frac{r_0}{1-\epsilon}. \tag{32.4}$$

Drawing exercise: Sketch an ellipse in the (x, y) plane, where θ is the angle with respect to the x-axis, and identify the distances $r_{\min}$ and $r_{\max}$ on your sketch.

Rewriting the orbit equation in terms of $x = r\cos\theta$,

$$\begin{aligned} r_0 &= r + \epsilon\, r\cos\theta && (32.5)\\ &= r + \epsilon\, x, && (32.6) \end{aligned}$$

after some re-arranging and squaring produces this:

$$(r_0 - \epsilon\, x)^2 = r^2 = x^2 + y^2, \tag{32.7}$$

$$r_0^2 = (1-\epsilon^2)x^2 + 2\epsilon r_0 x + y^2. \tag{32.8}$$

Completing the square (for the terms involving x) we obtain

$$(1-\epsilon^2)(x-x_0)^2 + y^2 = r_0^2 + (1-\epsilon^2)x_0^2, \tag{32.9}$$

where

$$x_0 = -\frac{\epsilon}{1-\epsilon^2} r_0. \tag{32.10}$$

The constants on the right hand side of (32.9) simplify to

$$r_0^2 + (1-\epsilon^2)x_0^2 = \left(1 + \frac{\epsilon^2}{1-\epsilon^2}\right) r_0^2 = \frac{r_0^2}{1-\epsilon^2}, \tag{32.11}$$

so that after multiplying through by $1-\epsilon^2$ we obtain:

$$(1-\epsilon^2)^2(x-x_0)^2 + (1-\epsilon^2)y^2 = r_0^2. \tag{32.12}$$

This we recognize as the high school formula for the ellipse,

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{32.13}$$

after defining major (a) and minor (b) axes as

$$a = \frac{r_0}{1-\epsilon^2}, \qquad b = \frac{r_0}{\sqrt{1-\epsilon^2}}. \tag{32.14}$$

As ϵ approaches 1 and

$$\frac{a}{b} = \frac{1}{\sqrt{1-\epsilon^2}} \to \infty, \tag{32.15}$$

the ellipse becomes very elongated. In that same limit

$$r_{\min} \to r_0/2, \qquad r_{\max} \to \infty. \tag{32.16}$$

The limiting orbit ($\epsilon = 1$) is a parabola.

32.1.3 Unbound orbits: the hyperbola

With just a few sign changes applied to the equations of the previous section we can establish that the orbits for the case $\epsilon > 1$ are hyperbolae. When the eccentricity crosses the threshold 1 the energy (32.3) becomes positive, thereby allowing the bodies to "escape to infinity"[1].

The constant

$$x_0 = \frac{\epsilon}{\epsilon^2 - 1} r_0 \tag{32.17}$$

is now positive and equation (32.12) should be written as

$$(\epsilon^2 - 1)^2 (x - x_0)^2 - (\epsilon^2 - 1) y^2 = r_0^2, \tag{32.18}$$

and motivate the definitions

$$a = \frac{r_0}{\epsilon^2 - 1}, \qquad b = \frac{r_0}{\sqrt{\epsilon^2 - 1}}, \tag{32.19}$$

so as to obtain

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{32.20}$$

This is the familiar equation of the hyperbola. Only one branch corresponds to the actual orbit. To determine which one, return to the orbit equation (32.1) before it was squared. Since $r_0 > 0$ because $A > 0$ for our (gravitational) attractive potential, the denominator $1 + \epsilon \cos\theta$ must be positive. The valid range of θ is therefore determined by

$$\cos\theta > -\frac{1}{\epsilon}. \tag{32.21}$$

The angles of the asymptotic straight trajectories of free motion far from the origin are therefore

$$\theta_\infty = \pm \arccos(-1/\epsilon). \tag{32.22}$$

By equation (32.20) the asymptotes do not pass through the origin but through the point $(x_0, 0)$.

Drawing exercise: Sketch the hyperbola (32.20) in the (x, y) plane, identifying the branch of the actual orbit, the linear asymptotes, and the point $(x_0, 0)$.

[1]This example of physics jargon could be the name of a rock band, or the title of a Hollywood movie.