Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 31: April 21

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

31.1 The gravitational two-body problem

In this lecture we solve, without approximations, the two-body problem for the case of the gravitational potential. The key steps in the solution are (1) using the conserved energy E, (2) replacing time by the orbit angle θ as the independent variable, and (3) replacing the radius by $u = 1/r$ as the dependent variable. Along the way we will see once again how the orbit closure property — the 2π periodicity of r with respect to θ — is unique to the inverse square force law.

31.1.1 Conserved energy

Here is the Lagrangian from lecture 29, written in terms of the variables r and θ of the separation $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ in the orbital plane:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{A}{r}, \qquad \mu = \frac{M_1 M_2}{M_1 + M_2}, \quad A = G M_1 M_2. \tag{31.1}$$

The Hamiltonian $\mathcal{H}$ has a constant value E since the time t is absent from $\mathcal{L}$. As the kinetic energy T is quadratic in the velocities, we know that $\mathcal{H} = T + V$ and therefore

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{r} \tag{31.2}$$

is the conserved energy. We can also use the fact (lecture 29) that the angular momentum

$$L_z = \mu r^2 \dot{\theta} \tag{31.3}$$

is conserved to write an expression for E that involves only r and $\dot{r}$:

$$E = \frac{1}{2}\mu\,\dot{r}^2 + \frac{L_z^2}{2\mu r^2} - \frac{A}{r}. \tag{31.4}$$

We could solve this equation for $\dot{r}$ and the resulting first order differential equation for $r(t)$. While this would express t as an explicit integral of a function of r, it is not obvious how that would explain the synchrony of the periodicity in r and θ. To address this mystery head-on, we will change the independent variable from t to θ.

31.1.2 Angle as independent variable

Using the conserved angular momentum, the time derivative of r takes the following form:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{L_z}{\mu r^2}. \tag{31.5}$$

Substituting this into (31.4) we obtain the following equation for the energy:

$$E = \frac{L_z^2}{2\mu r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{L_z^2}{2\mu r^2} - \frac{A}{r}. \tag{31.6}$$

This again is a first order differential equation for r that we could solve such that θ is expressed as an explicit integral of a function of r. And again, it is not at all obvious why the corresponding inverse function (r as a function of θ) would have periodicity 2π. To overcome this difficulty we make a change of the dependent variable. Not all the steps in our solution have a clear motivation, and this is one of them.

31.1.3 Inverse distance as dependent variable

Defining a new dependent variable by

$$u(\theta) = 1/r(\theta), \tag{31.7}$$

we obtain the following expression for the θ-derivative term in (31.6):

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}, \qquad \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2. \tag{31.8}$$

The resulting expression for E has terms that are either quadratic or linear in u:

$$E = \frac{L_z^2}{2\mu}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - Au. \tag{31.9}$$

We've saved the best trick for last. The conservation of energy, with the change of independent variable,

$$0 = \frac{dE}{dt} = \frac{dE}{d\theta}\frac{d\theta}{dt}, \tag{31.10}$$

implies

$$0 = \frac{dE}{d\theta} \tag{31.11}$$

except possibly when $\dot{\theta} = 0$. However, for the type of orbit we will mostly be interested in this never happens and we can therefore impose property (31.11) on equation (31.9):

$$0 = \frac{dE}{d\theta} = \frac{L_z^2}{2\mu}\left(2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta}\right) - A\frac{du}{d\theta}. \tag{31.12}$$

The whole point of using the inverse distance u is evident now, because all three terms have $du/d\theta$ as a common factor, which when it is divided out reduces the quadratic (in u) terms into linear ones and the linear term into a term that doesn't involve u at all. Here is the resulting linear equation for u:

$$\frac{d^2u}{d\theta^2} + u = \frac{A\mu}{L_z^2}. \tag{31.13}$$

This must be satisfied at all θ where $du/d\theta \neq 0$. The exceptions, where u (and also r) is a local maximum or minimum, are isolated points and play no role in determining the function $u(\theta)$[1].

[1] This is a result from analysis (math), regarding the analyticity of solutions of differential equations.

31.1.4 Orbit closure

The most general solution of (31.13) is the sum of an arbitrary solution to the inhomogeneous equation, $u = A\mu/L_z^2$, and the most general solution to the homogeneous equation:

$$u(\theta) = \frac{A\mu}{L_z^2} + u_0 \cos(\theta - \theta_0). \tag{31.14}$$

This already confirms that orbits are closed: as θ ranges over a full period, 0 to 2π, the radius $r(\theta) = 1/u(\theta)$ returns to its value at the start of the orbit. Now is as good a time as any to see how this seemingly mundane fact is closely tied to the inverse square law.

Suppose the force of gravity was inverse-cube instead of inverse-square. Letting T be the kinetic energy which is unchanged, the expression for the total energy is changed to

$$\begin{aligned} E &= T - \frac{A'}{r^2} && (31.15) \\ &= T - A'u^2, && (31.16) \end{aligned}$$

where A' is the analog of $A = GM_1M_2$ for the inverse-cube law. Everything in our solution of the differential equation for u would go through exactly as before, except that there will not be a constant term (new factor shown in parentheses):

$$\frac{d^2u}{d\theta^2} + u = (2u)\frac{A'\mu}{L_z^2}. \tag{31.17}$$

$$\frac{d^2u}{d\theta^2} - K^2 u = 0, \qquad K = \sqrt{\frac{2A'\mu}{L_z^2} - 1}. \tag{31.18}$$

We have assumed in the definition of K that $L_z^2 < 2A'\mu$ (small angular momentum). The most general solution to this differential equation is

$$u(\theta) = u_0 \cosh K(\theta - \theta_0). \tag{31.19}$$

Clearly $r(\theta) = 1/u(\theta)$ does *not* have periodicity 2π and instead describes the double-death-spiral orbit shown below[2].

31.1.5 The orbit equation

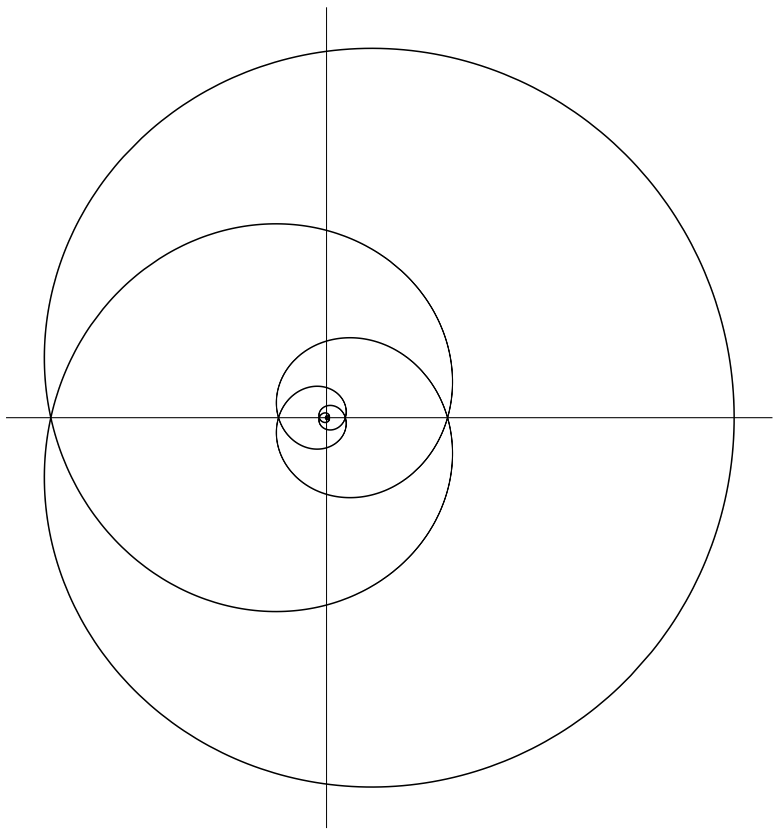
Returning to the inverse square law, we finish by writing the solution for the orbit in a standard form with some definitions. The constant term defines $1/r_0$, the angular average of the inverse distance:

$$u(\theta) = \frac{1}{r(\theta)} = \frac{1}{r_0}\left(1 + \epsilon\cos(\theta - \theta_0)\right), \qquad r_0 = \frac{L_z^2}{A\mu}, \quad \epsilon = u_0 r_0. \tag{31.20}$$

Since u_0 was arbitrary, we absorb it in the definition of the arbitrary dimensionless parameter ϵ, the "eccentricity" of the orbit. The standard form of the orbit equation is now

$$r(\theta) = \frac{r_0}{1 + \epsilon\cos(\theta - \theta_0)}. \tag{31.21}$$

[2] Yet further evidence of the existence of God (orbits of the true law of gravity do not do this).



An orbit (31.19) for inverse-cube gravity when the angular momentum is such that $K = 0.3$.