Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 30: April 19

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

30.1 The two-body problem

30.1.1 Circular orbits

Let's take as our two-body potential the gravitational energy of two point masses:

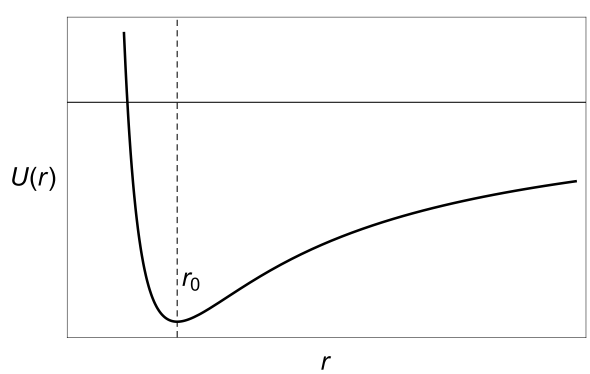
$$V(r) = -\frac{A}{r}, \qquad A = GM_1M_2. \tag{30.1}$$

Using the identity

$$\frac{L_z^2}{\mu r^3} = -\frac{d}{dr}\left(\frac{L_z^2}{2\mu r^2}\right) = -\frac{d}{dr}\left(\frac{B}{r^2}\right), \qquad B = \frac{L_z^2}{2\mu}, \tag{30.2}$$

we can write the equation of motion for the relative distance in terms of a single "effective potential":

$$\mu\ddot{r} = -\frac{dU}{dr}, \qquad U(r) = \frac{B}{r^2} - \frac{A}{r}. \tag{30.3}$$



The attractive A term, due to gravity, dominates at large r while the repulsive B term, called the "centrifugal barrier", dominates at small r. When $L_z = 0$ there is no barrier and there is nothing to prevent the two masses from falling into each other.

Assuming $L_z \neq 0$, the radius r_0 defined by

$$\left.\frac{dU}{dr}\right|_{r_0} = 0, \tag{30.4}$$

is special. When the initial conditions are

$$\begin{aligned} r(0) &= r_0 && (30.5)\\ \dot{r}(0) &= 0, && (30.6) \end{aligned}$$

then the equation of motion

$$\mu\ddot{r}(0) = -\left.\frac{dU}{dr}\right|_{r_0} = 0 \qquad (30.7)$$

implies $r(t)$ will stay constant at the special radius r_0. This corresponds to a circular orbit. For this to happen the separation must be at the minimum of the effective potential:

$$0 = \left.\frac{dU}{dr}\right|_{r_0} = -\frac{2B}{r_0^3} + \frac{A}{r_0^2}, \qquad (30.8)$$

$$r_0 = \frac{2B}{A} = \frac{L_z^2/\mu}{GM_1M_2}. \qquad (30.9)$$

Since both L_z and $r = r_0$ are constant for circular orbits,

$$\dot{\theta} = \frac{L_z}{\mu r_0^2} = \omega_\theta = \text{constant}. \qquad (30.10)$$

We can use this in equation (30.9),

$$r_0 = \frac{\mu}{GM_1M_2}\left(\frac{L_z}{\mu}\right)^2 = \frac{\mu}{GM_1M_2}\left(\omega_\theta\, r_0^2\right)^2, \qquad (30.11)$$

which can be written compactly as

$$G(M_1 + M_2) = \omega_\theta^2\, r_0^3. \qquad (30.12)$$

This is Kepler's famous 1-2-3 law[1] for the special case of circular orbits. Later we will see how this generalizes to general, elliptical orbits.

30.1.2 Perturbation of the circular orbit

Let's study what happens when the circular orbit is slightly perturbed. We will assume the perturbation δr in the separation is so slight that we can approximate the function $U(r)$ near its minimum by a parabola,

$$\begin{aligned} U(r) &\approx U(r_0) + \frac{1}{2}U''(r_0)(r - r_0)^2 && (30.13)\\ &= U_0 + \frac{1}{2}K\delta r^2, && (30.14) \end{aligned}$$

where by using (30.9) the "stiffness" is given by

$$K = \frac{6B}{r_0^4} - \frac{2A}{r_0^3} = \frac{3A}{r_0^3} - \frac{2A}{r_0^3} = \frac{A}{r_0^3}. \qquad (30.15)$$

The equation of motion for the perturbation

$$\mu\,\ddot{\delta r} = -\frac{dU}{dr} = -K\delta r, \qquad (30.16)$$

[1] This is my name for the law. Kepler did not have access to information about the "1" in mass^1 $\times$ period^2 $\propto$ radius^3.

is that of a harmonic oscillator with frequency

$$\omega_r = \sqrt{K/\mu}. \tag{30.17}$$

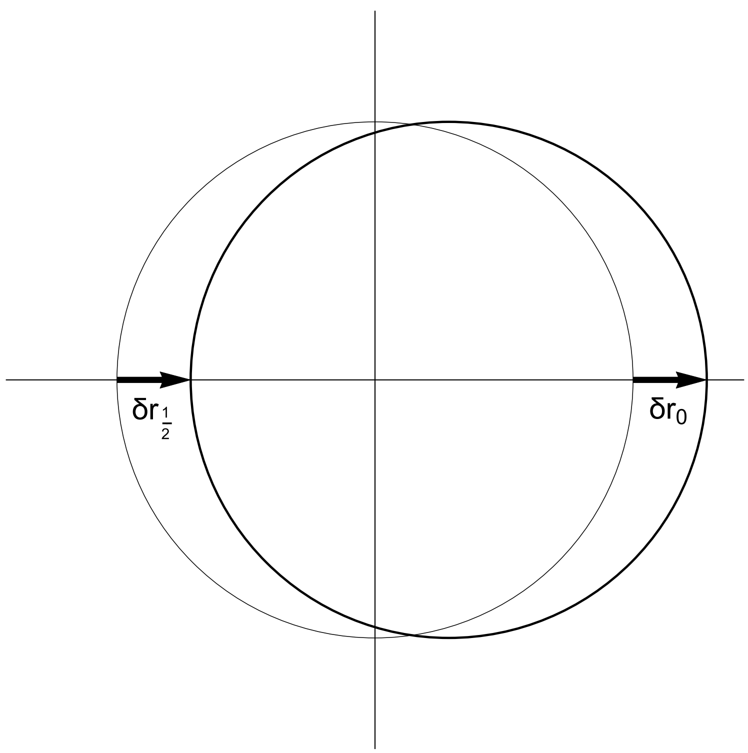
The subscript reminds us these are oscillations in the distance r between the bodies. In our perturbed-periodic orbital motion there is another frequency: the angular frequency of orbit completion ω_θ. Comparing these by expressing them in terms of a common set of parameters we find:

$$\omega_r^2 = K/\mu = \frac{A}{\mu r_0^3} = \frac{G(M_1+M_2)}{r_0^3} \tag{30.18}$$

$$\omega_\theta^2 = \frac{G(M_1+M_2)}{r_0^3} \qquad \text{(Kepler)}. \tag{30.19}$$

The time taken to complete one radial oscillation is therefore equal to the time taken to complete one orbit. Note that for the perturbed orbit only L_z is strictly constant, not $\dot{\theta} = \omega_\theta$. A better statement of the result of our analysis is that the period T of the two kinds of motion is the same in the limit of small perturbations.

The apparent coincidence that $\omega_r = \omega_\theta$ is consistent with the possibility of closed *elliptical orbits*, that is, orbits with an apogee — point of greatest distance from the center of force — that does not precess in angle.



Perturbation of a circular orbit of period T showing the maximum positive amplitude radial perturbation δr_0 at time $t = 0$ and the maximum negative amplitude perturbation $\delta r_{1/2}$ at time $t = T/2$.

In your homework assignment you will discover that the absence of precession is very special to the inverse-square law of attraction between point masses and is upset by even small deviations from this law.