Lectures 30 & 31

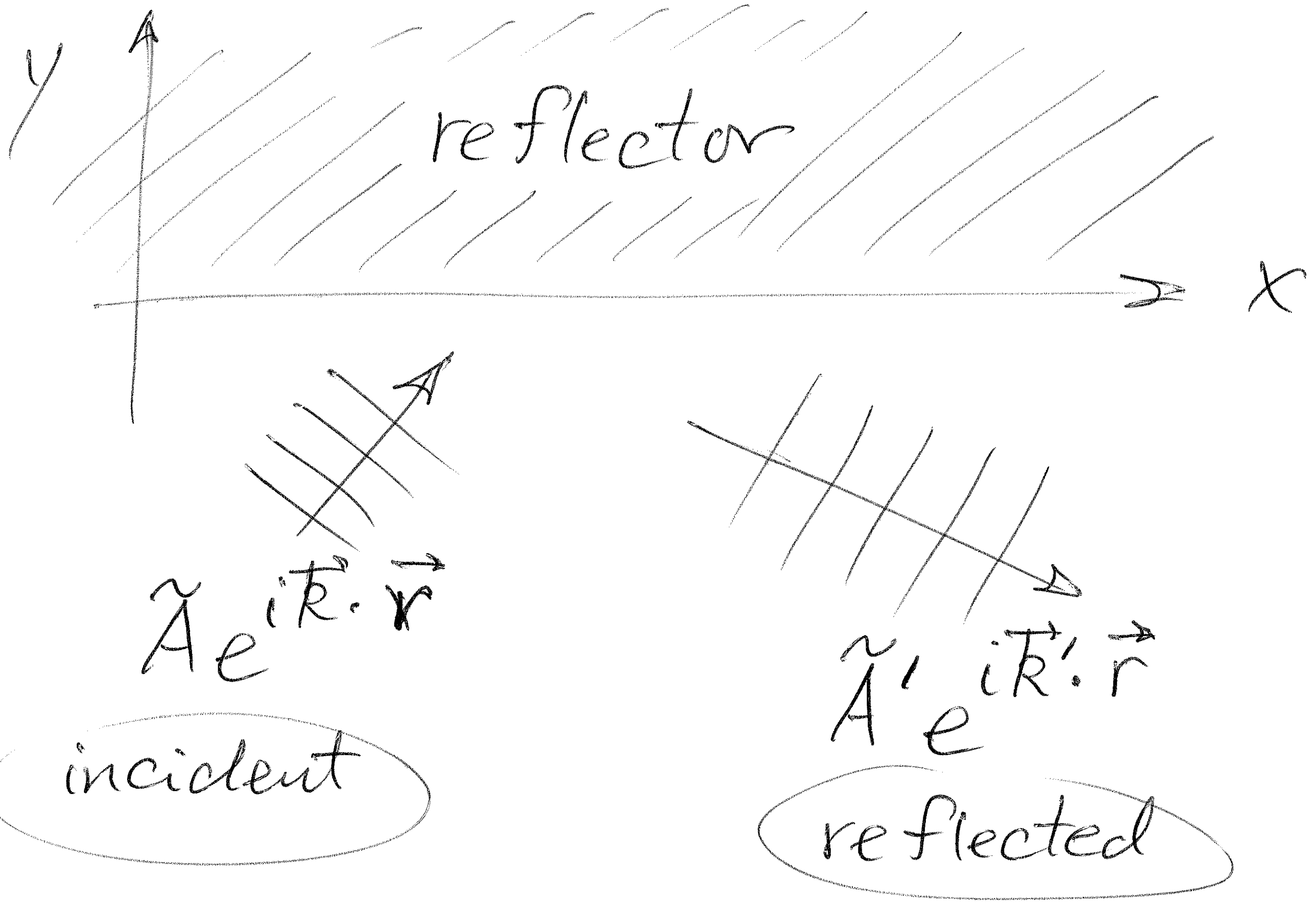
Reflection and transmission/refraction of waves at interfaces

In all the phenomena we study at interfaces we imagine that waves of a single frequency ω have been incident for a long time so "transient effects" have decayed and the reflected & transmitted waves also have frequency ω.

Reflection when the interface forces the amplitude to be zero

This happens in the case of EM waves incident on a good conductor. Recall the property that there can be no static $\vec{E}$ field inside a conductor. Something close to this is true even when $\vec{E}$ is oscillating in time (Phys 3327).

Since the time dependence of our waves is always $e^{-i\omega t}$ we use the compact plane-wave notation introduced earlier: 31.3



- equal freq. $\Rightarrow |\vec{k}| = |\vec{k}'|$

- $\tilde{\Psi}(x,0) = 0 \Rightarrow$

$$\tilde{A}e^{ik_x x} + \tilde{A}'e^{ik'_x x} = 0$$

for all x

(31.4)

The two phasors in this equation rotate at different rates with x if $k_x \neq k_x'$, and the equation would not be true for all x (except in the trivial case $\tilde{A} = \tilde{A}' = 0$). We therefore have $\underline{k_x = k_x'}$. From this we get

$$(\tilde{A} + \tilde{A}')e^{ik_x x} = 0, \quad \text{all } x$$

and so

$$\tilde{A} + \tilde{A}' = 0.$$

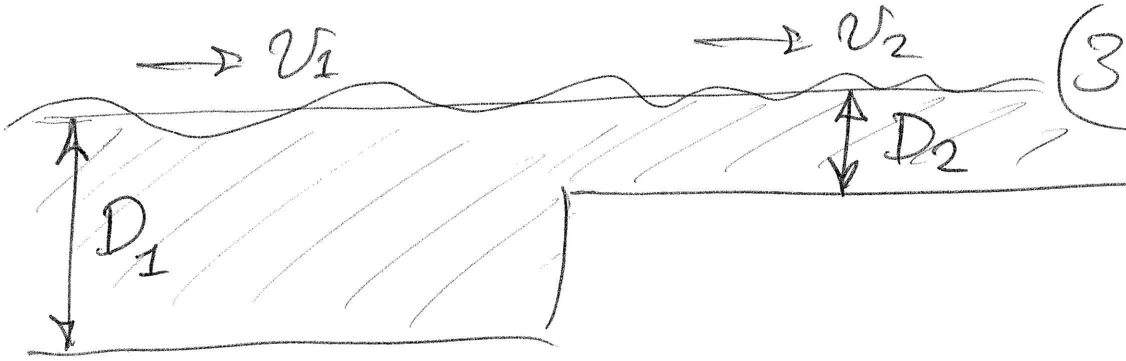
- Since the magnitudes and x-components of $\vec{k}$ and $\vec{k}'$ are equal, their y-components must have the same magnitude. But only the incident wave may have $k_y > 0$ (there is only one source at $y = -\infty$), so $k_y' = -k_y$.

In Quantum Mechanics particles are described by waves, and $\hbar\vec{k}$ is their momentum. The result $k_x = k_x'$ therefore says that the x-component of momentum is conserved. This is reasonable considering our interface, which has no structure along x and hence cannot produce forces in that direction.

Normal incidence on interface separating media with different wave speeds

Image surface waves originating far out in the ocean and arriving at a continental "shelf" where the depth is less. Will waves be transmitted into the shallow part? Will there be a reflected wave?

(31.7)



In the limit wavelength much longer than the depth, "shallow" surface waves have this dispersion relation:

$$\omega = \sqrt{gD}\, k \quad , \quad D = \text{depth}$$

For fixed ω in both depths we have

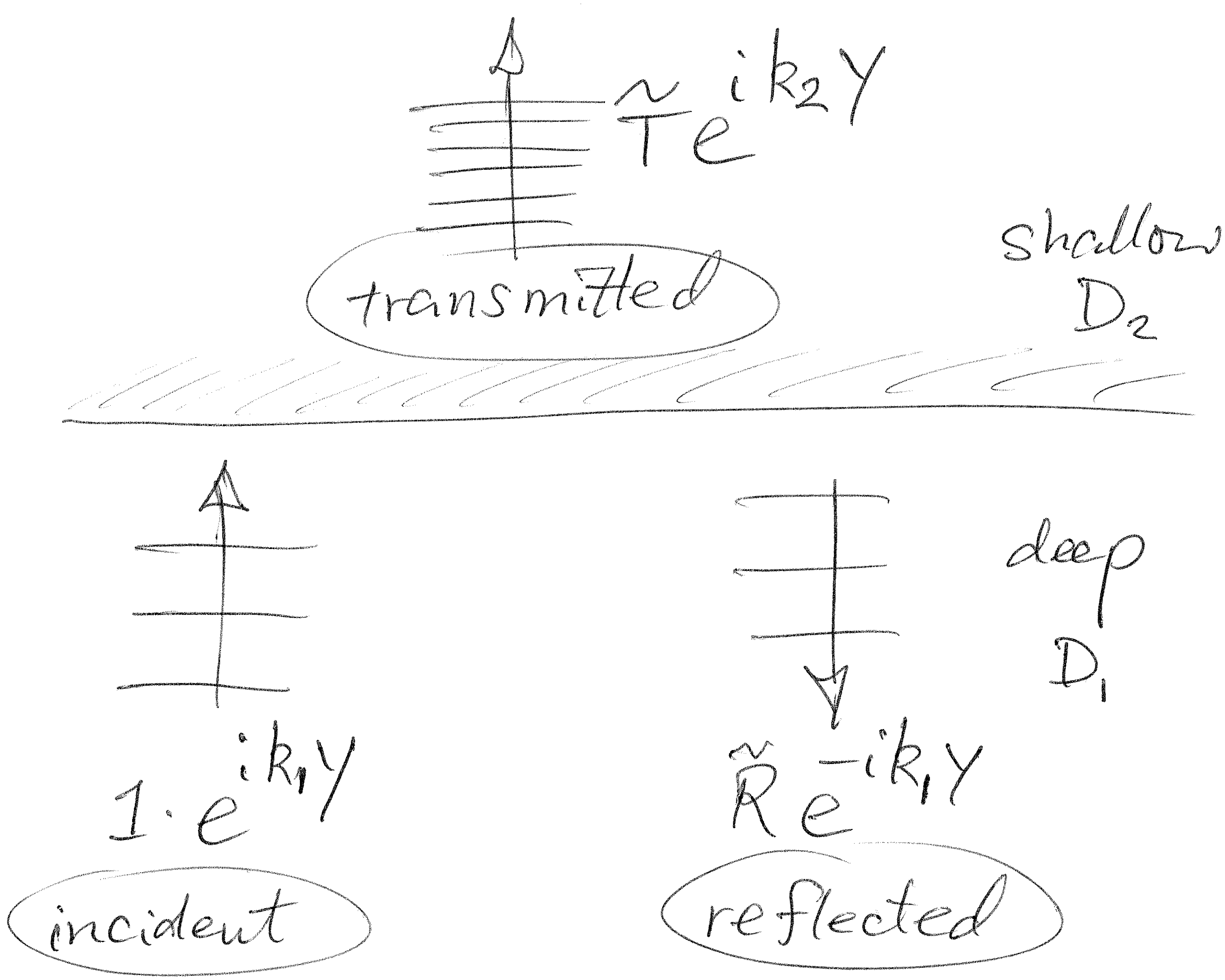
$$\omega = v_1 k_1 = v_2 k_2$$

where $v_1/v_2 = \sqrt{D_1/D_2}$.

So waves in the shallow

water move more slowly and have larger k (shorter wavelength).

Top view:



Given k_1 we find k_2 using

$$v_1 k_1 = v_2 k_2 .$$

All that remains is to (31.9)
determine the amplitudes
$\tilde{R}$ & $\tilde{T}$ (We've set the
incident amplitude equal to
1 without loss of generality; for
any other incident amplitude
$\tilde{A}$, just multiply our results for
1 as incident: $\tilde{R} \to \tilde{A}\tilde{R}$, $\tilde{T} \to \tilde{A}\tilde{T}$)

To determine $\tilde{R}$ & $\tilde{T}$ we need
two equations. These are that
the wave amplitudes "match" at
the interface, where "match"
means 1) Φ is continuous

2) $\frac{d\Phi}{dy}$ is continuous

(31.10)

1) $\Rightarrow \quad 1+\tilde{R} = \tilde{T}$

($y=0$ limit of $e^{ik_1 y} + \tilde{R}e^{-ik_1 y} = \tilde{T}e^{ik_2 y}$)

2) $\Rightarrow \quad ik_1 - ik_1\tilde{R} = ik_2\tilde{T}$

($y=0$ limit of $\frac{d}{dy}\left(e^{ik_1 y} + \tilde{R}e^{-ik_1 y}\right) = \frac{d}{dy}\left(\tilde{T}e^{ik_2 y}\right)$),

Solving the two equations in the unknowns $\tilde{R}, \tilde{T}$:

$$ik_1(1-\tilde{R}) = ik_2(1+\tilde{R})$$

$$i(k_1-k_2) = i(k_2+k_1)\tilde{R}$$

$$\Rightarrow \quad \tilde{R} = \frac{k_1-k_2}{k_1+k_2}$$

note that $|\tilde{R}| < 1$.

$$\tilde{T} = \frac{k_1 + k_2}{k_1 + k_2} + \frac{k_1 - k_2}{k_1 + k_2}$$

$$\tilde{T} = \frac{2k_1}{k_1 + k_2}$$

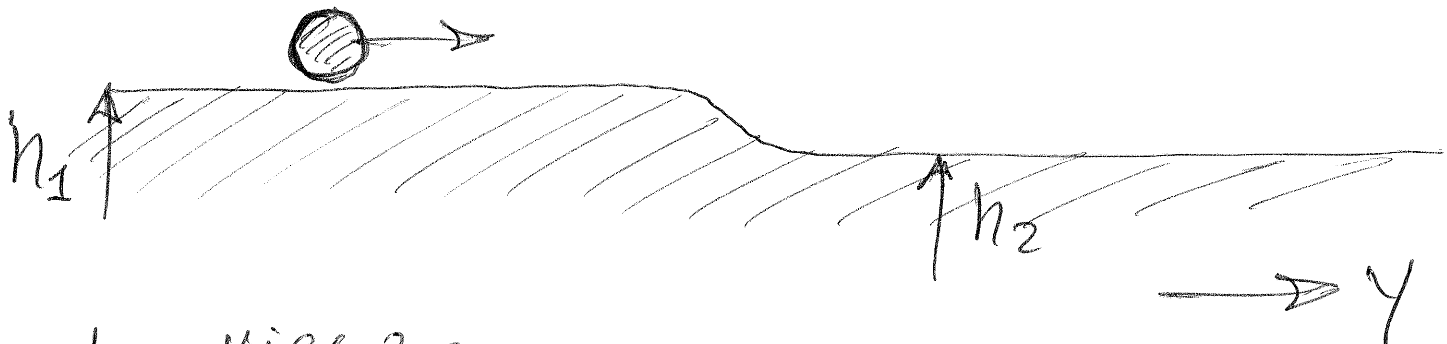
- We get strong reflection, i.e. $|\tilde{R}| \simeq 1$, if either $k_1 \gg k_2$ or $k_1 \ll k_2$.

- There is poor transmission, i.e. $\tilde{T} \to 0$, if $k_1 \ll k_2$

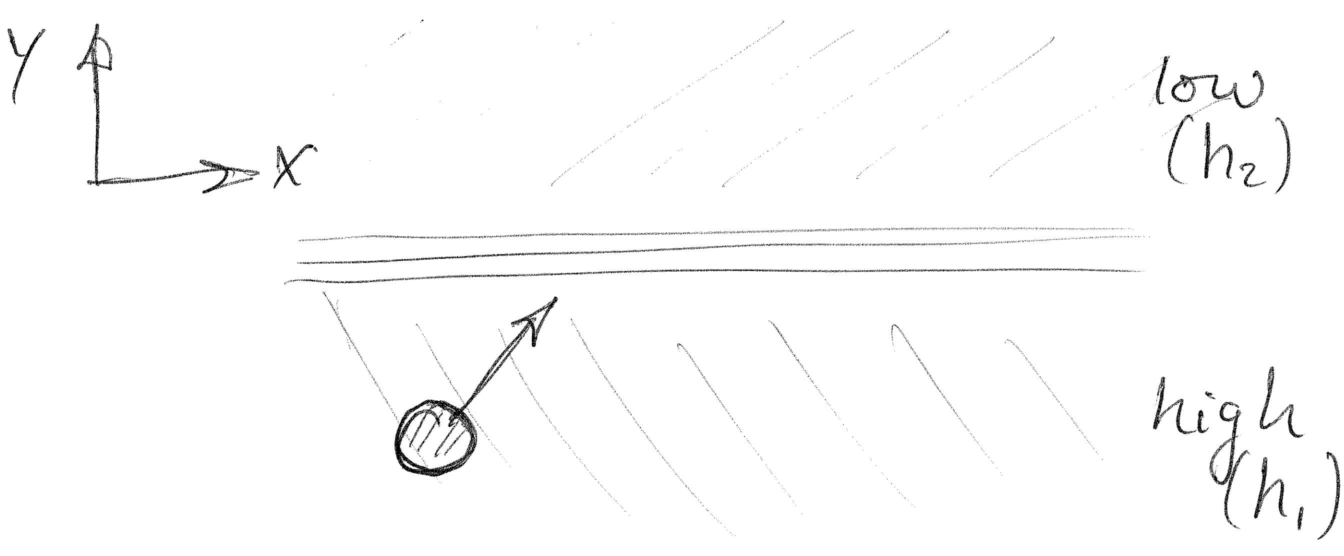
Do these results make sense in the context of waves incident from deep/shallow water?

Non-normal incidence: refraction

As a warm-up problem, consider what happens when a marble rolls from one flat surface to another, at a different height:

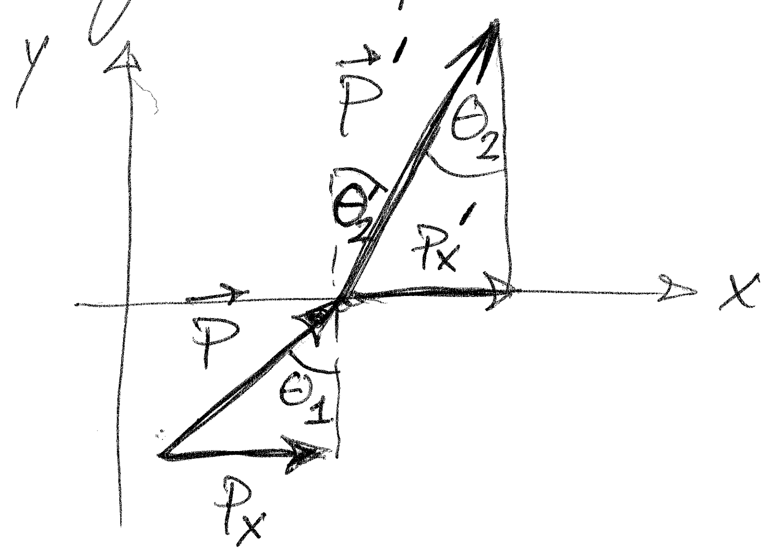


top view:



How do we determine the marble trajectory in the x-y plane using mechanics? (31.13)

The x-component of the marble's momentum is conserved, and we can work out what happens to the y-component using energy conservation:



$$P_x = P_x'$$

or

$$P\sin\theta_1 = P'\sin\theta_2$$

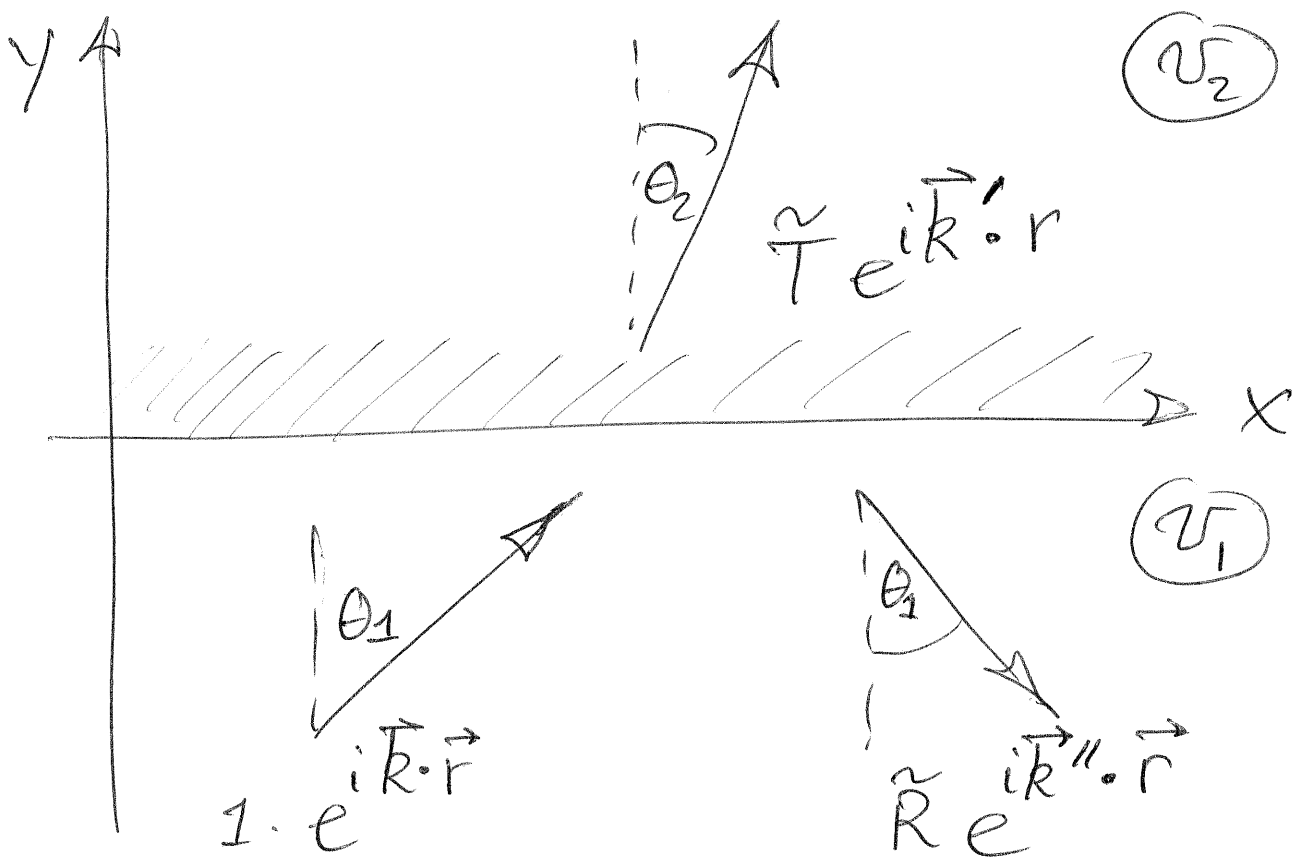
energy conservation:

$$\frac{P^2}{2m} + mgh_1 = \frac{P'^2}{2m} + mgh_2$$

$$h_1 > h_2 \Rightarrow P' > P \Rightarrow \theta_2 < \theta_1$$

The situation with waves is almost the same. The x-components of the wave vectors on both sides of the interface are the same and the y-components are found from the property that the wave has frequency ω everywhere ("energy conservation" = "no transient behavior"). There is one difference however: there is a reflected wave as well, and the amplitudes $\tilde{R}$ & $\tilde{T}$ of the reflected and transmitted (refracted) waves have to be worked out. But that very thing applies to the marble as well, when it is described as a quantum wave! (31.15)

We will skip the calculation of the amplitudes $\tilde{R}$ & $\tilde{T}$ and focus just on the wave-vectors:



- $k_x = k'_x = k''_x$

- $\underbrace{|\vec{k}|\sin\theta_1}_{k_x} = \underbrace{|\vec{k}'|\sin\theta_2}_{k'_x}$

- $\omega = v_1 |\vec{k}| = v_1 |\vec{k}''| = v_2 |\vec{k}'|$

- $|\vec{k}| = |\vec{k}''|$

- $k_y'' = -k_y$

- $|\vec{k}| \sin\theta_1 = |\vec{k}'| \sin\theta_2$

$$\Downarrow$$

$$\frac{\omega}{v_1} \sin\theta_1 = \frac{\omega}{v_2} \sin\theta_2$$

In the case of light, the wave ("phase") velocity is written

$$v_1 = \frac{c}{n_1}, \quad v_2 = \frac{c}{n_2}$$

n_1 & n_2 = indices of refraction in the two media

Hence: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (Snell)