Physics 3318: Analytical Mechanics	**Spring 2017**

Lecture 3: February 1

Instructor: Veit Elser

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

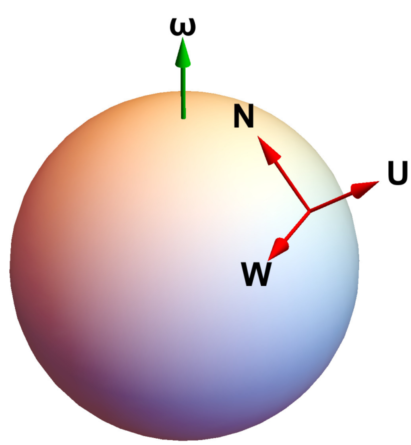
3.1 Foucault pendulum

In lecture 2 we derived the fictitious forces "as seen by an observer on a rotating body":

$$\mathbf{F}_{\text{fict}}/m = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \mathring{\mathbf{r}} - \dot{\boldsymbol{\omega}} \times \mathbf{r}. \tag{3.1}$$

Writing this in the form of a fictitious acceleration (force divided by the particle's mass) reminds us that these forces are purely kinematical artifacts, and are not caused by the mass (such as the force of gravity). In this lecture we use fictitious forces to explain the rotation of the plane of oscillation of the Foucault pendulum.

Below is a diagram of the frame we would like to use to describe the pendulum: an up basis vector **U** normal to the surface of the earth at the site of the pendulum, and orthogonal vectors **N** and **W** that point north and west respectively.



Question: What rule can you use to establish that the direction of $\mathbf{W}$ (west) is correct?

A problem with our choice of frame is that its origin does not lie on the axis of rotation through the center of the earth — as it should. We would have to include additional fictitious forces if we allowed our non-inertial frame to experience linear accelerations as well as rotations. To avoid this complication, imagine translating the frame as shown to the center of the earth, so its origin passes through the axis of rotation. In our analysis of the pendulum, or any other motion near the surface of the earth, we have to remember to add the radius R of the earth to the coordinate along the $\mathbf{U}$ basis vector.

We also switch to the standard names of the basis vectors: $\mathbf{U} = \hat{\mathbf{z}}$, $\mathbf{N} = \hat{\mathbf{x}}$, $\mathbf{W} = \hat{\mathbf{y}}$. *Note: because everything in this lecture will be "as seen in the body frame", we dispense with primes on coordinates and basis vectors.*

Estimation exercise: For a pendulum of length $l = 1\,\text{m}$, compare the relative magnitudes[1] of the centrifugal and Coriolis forces. In your estimate you can ignore the average value of a force since that only affects the equilibrium position and not the actual motion.

[1] In order-of-magnitude estimates you may round to the nearest power of 10: $\pi \approx 1$, $2^5 \approx 100$.

Using elementary geometry we can express $\boldsymbol{\omega}$ in terms of the basis vectors:

$$\boldsymbol{\omega} = \Omega(\cos\lambda\,\hat{\mathbf{x}} + \sin\lambda\,\hat{\mathbf{z}}) \qquad \Omega = \frac{2\pi}{24\,\text{hours}}. \tag{3.2}$$

We'll assume small amplitude oscillations of the pendulum mass m and neglect the z component of velocity in the Coriolis force relative to the x and y components:

$$\begin{aligned}
\mathbf{F}_{\text{cor}} &= -2m\,\boldsymbol{\omega}\times\mathbf{v} && (3.3)\\
&\approx -2m\,\boldsymbol{\omega}\times(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}) && (3.4)\\
&= -2m\Omega\left(\cos\lambda\,v_y\hat{\mathbf{z}} + \sin\lambda\,v_x\hat{\mathbf{y}} - \sin\lambda\,v_y\hat{\mathbf{x}}\right). && (3.5)
\end{aligned}$$

Because we are interested mostly in the motion in the x-y plane (the plane tangent to the earth at the pendulum), we restrict ourselves to the equations for $\ddot{x}$ and $\ddot{y}$. Gravity provides the "true" restoring force:

$$\mathbf{F}_g = -m\omega_g^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) \qquad \omega_g = \sqrt{g/l}. \tag{3.6}$$

Question: What approximation was made in the equation for the gravity force?

The net x and y components of force — true gravity plus fictitious Coriolis — acting on the pendulum mass in the small amplitude approximation is therefore

$$\begin{aligned}
F_x &= -m\omega_g^2\,x + 2m\Omega\sin\lambda\,\dot{y} && (3.7)\\
F_y &= -m\omega_g^2\,y - 2m\Omega\sin\lambda\,\dot{x}. && (3.8)
\end{aligned}$$

Now that we have worked out the physics, the rest is just math. The equations of motion are the following system of linear differential equations,

$$\begin{aligned}
\ddot{x} &= -\omega_g^2\,x + 2\omega_p\,\dot{y} && (3.9)\\
\ddot{y} &= -\omega_g^2\,y - 2\omega_p\,\dot{x}, && (3.10)
\end{aligned}$$

where we have defined a latitude-dependent "precession frequency"

$$\omega_p = \Omega\sin\lambda \tag{3.11}$$

whose interpretation will be clear shortly.

Question: Relate the magnitudes of the Coriolis and gravity force terms to the ratio ω_p/ω_g.

To solve the two equations we use the trick that these are the real and imaginary parts of the following single equation for $z = x + \mathrm{i}y$:

$$\overset{\circ\circ}{z} = -\omega_g^2\, z - \mathrm{i}2\omega_p\, \overset{\circ}{z}. \tag{3.12}$$

Basic solutions of this (linear, homogeneous) differential equation are $z = e^{\mathrm{i}\alpha t}$ where α satisfies the algebraic equation

$$-\alpha^2 = -\omega_g^2 + 2\omega_p\alpha. \tag{3.13}$$

Since $\omega_g \gg \omega_p$, the solution to 0^{th} order in the small quantity ω_p is

$$\alpha_0 = \pm\omega_g. \tag{3.14}$$

Substituting this 0^{th} order α into the small term of (3.13) we get the following approximation for α:

$$-\alpha^2 = -\omega_g^2 \pm 2\omega_p\omega_g \tag{3.15}$$

$$\alpha^2 = \omega_g^2(1 \mp 2\omega_p/\omega_g) \tag{3.16}$$

$$\begin{aligned} \alpha &\approx \pm\omega_g(1 \mp \omega_p/\omega_g) && (3.17)\\ &= \pm\omega_g - \omega_p. && (3.18) \end{aligned}$$

Combining the two basic solutions with equal amplitudes we get the following particular solution:

$$\begin{aligned} z &= A\left(e^{\mathrm{i}(\omega_g-\omega_p)t} + e^{\mathrm{i}(-\omega_g-\omega_p)t}\right) && (3.19)\\ &= \left(2Ae^{-\mathrm{i}\omega_p t}\right)\cos\omega_g t. && (3.20) \end{aligned}$$

The point z oscillates along the axis defined by $e^{-\mathrm{i}\omega_p t}$ in the complex plane with the oscillation frequency of the ordinary pendulum, ω_g. Since $e^{-\mathrm{i}\omega_p t}$ rotates as well, albeit very slowly, we identify ω_p as the precession rate of the pendulum's plane of oscillation.

Question: Why should you be very skeptical of a science museum in Quito advertising a Foucault pendulum?

Question: The sense of precession is clockwise in the northern hemisphere, since $\omega_p > 0$ when $\lambda > 0$. Give an easy way to check this conclusion.

Question: It's clear that combining the two basic solutions with amplitudes having different phases (*e.g.* opposite instead of equal signs) is equivalent to changing the origin of time. But what if the magnitudes of the amplitudes were different, say (extreme case) $z = Ae^{\mathrm{i}(\omega_g-\omega_p)t}$? What would the Foucault pendulum be doing in that case?