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29.1 The two-body problem

29.1.1 Reducing six degrees of freedom to one

Science is often evaluated by its ability to predict, and seen through that lens the history of classical mechanics is the history of predicting the orbits of moons and planets to ever greater precision. An important milestone in that history was the complete solution of the motion of two bodies attracted by gravity. Although the bodies might be stars or planets, we assume their separation is so much greater than their sizes that we may represent them as points whose only relevant physical characteristic is their mass. The only parameters in the two-body problem are therefore the two masses M_1 and M_2.

We will completely solve the gravitational two-body problem in the Lagrangian formalism. As there are no constraints on the Cartesian coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the bodies, we write the kinetic and potential energies of the Lagrangian in terms of them:

$$\mathcal{L} = \frac{1}{2}M_1\,\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_1 + \frac{1}{2}M_2\,\dot{\mathbf{r}}_2\cdot\dot{\mathbf{r}}_2 - V(|\mathbf{r}_1-\mathbf{r}_2|). \tag{29.1}$$

We keep the functional form of the potential arbitrary for now, but specialized to the case where it only depends on the distance between the bodies. Our system presently has six degrees of freedom.

The structure of $\mathcal{L}$ simplifies when we make a linear change of variables to a new set of generalized coordinates. From experience we know that the motion of the system's center of mass is very simple for an isolated system such as ours, so that will be one of our new coordinates:

$$\mathbf{R} = \frac{M_1\mathbf{r}_1 + M_2\mathbf{r}_2}{M_1+M_2}. \tag{29.2}$$

As the potential energy depends only on the difference of positions, we let that define the other coordinate:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{29.3}$$

Expressing $\mathbf{r}_1$ and $\mathbf{r}_2$ (which appear individually in the kinetic energy) in terms of $\mathbf{R}$ and $\mathbf{r}$ is a routine exercise in algebra:

$$\mathbf{r}_1 = \mathbf{R} + \left(\frac{M_2}{M_1+M_2}\right)\mathbf{r} \tag{29.4}$$

$$\mathbf{r}_2 = \mathbf{R} - \left(\frac{M_1}{M_1+M_2}\right)\mathbf{r}. \tag{29.5}$$

The Lagrangian, when expressed in terms of the new variables, takes the form

$$\mathcal{L} = \frac{1}{2}(M_1 + M_2)\,\dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2}\mu\,\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - V(|\mathbf{r}|), \tag{29.6}$$

where

$$\mu = \frac{M_1 M_2}{M_1 + M_2} \tag{29.7}$$

is called the "reduced mass". The term "reduced" is a reference to the fact that when applied to a system comprising very different masses, say $M_1 \ll M_2$, then by equation (29.7) μ is a slight reduction of the mass M_1 of the lighter body.

Since the center-of-mass position $\mathbf{R}$ does not appear in $\mathcal{L}$, its conjugate momentum,

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{R}}} = (M_1 + M_2)\dot{\mathbf{R}} \tag{29.8}$$

is conserved. This is just the total momentum of the system (as defined in freshman mechanics) and its constancy implies $\mathbf{R}$ moves linearly and uniformly in time. The interesting part of the two-body problem is now reduced to effectively the one-body problem associated with the relative position degree of freedom $\mathbf{r}$.

Here are the Euler-Lagrange equations for the relative position:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}}\right) &= \frac{\partial \mathcal{L}}{\partial \mathbf{r}} && (29.9) \\ \mu\ddot{\mathbf{r}} &= -V'(|\mathbf{r}|)\,\hat{\mathbf{r}}. && (29.10) \end{aligned}$$

Here the symbol V' is the derivative of the potential function with respect to its scalar argument, and $\hat{\mathbf{r}} = \partial|\mathbf{r}|/\partial\mathbf{r}$ is the unit vector parallel to $\mathbf{r}$. By (29.10) the one-body problem is therefore that of a body of mass μ at position $\mathbf{r}$ subject to the central force

$$\mathbf{F} = -V'(|\mathbf{r}|)\hat{\mathbf{r}}. \tag{29.11}$$

The vector $\mathbf{r}$ and its velocity $\dot{\mathbf{r}}$ define a plane through the origin with normal vector $\mathbf{r} \times \dot{\mathbf{r}}$. This normal vector, and hence the plane it defines, is constant in time

$$\frac{d}{dt}\left(\mathbf{r} \times \dot{\mathbf{r}}\right) = \mathbf{r} \times \ddot{\mathbf{r}} = 0 \tag{29.12}$$

since by (29.10) $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$ (the defining property of a "central force"). Therefore, since $\mathbf{r}$ always moves within a particular plane, from now on we define $\mathbf{r}$ as the position vector in this two-dimensional (x, y) plane. This has the effect of reducing the two-body problem to two degrees of freedom.

Neglecting the constant center-of-mass kinetic energy, we rewrite the Lagrangian (29.6) in terms of polar coordinates for the two-dimensional position $\mathbf{r}$:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{29.13}$$

As the coordinate θ is absent, the conjugate momentum

$$L_z = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2\dot{\theta} \tag{29.14}$$

is constant in time. The notation for this conserved quantity reminds us this is angular momentum in the one-body problem.

Question: Is L_z also the angular momentum about the center of mass in the original two-body problem?

We have now succeeded in reducing the motion of six degrees of freedom to the motion of a single degree of freedom, the distance between the bodies. Starting with the Euler-Lagrange equation,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) &= \frac{\partial \mathcal{L}}{\partial r} && (29.15) \\ \mu \ddot{r} &= \mu r \dot{\theta}^2 - V'(r) && (29.16) \end{aligned}$$

we then replace $\dot{\theta}$ by its value from the conserved angular momentum (29.14):

$$\mu \ddot{r} = \frac{L_z^2}{\mu r^3} - V'(r). \tag{29.17}$$

We will study this equation in detail, in particular, to establish the elliptic/hyperbolic shapes of orbits when V is the gravitational potential function.