Lecture 28

An infinite straight wire carrying current $I$ is the source of magnetic field that encircles the wire in a right-handed sense (by definition) and has magnitude $B = \frac{\mu_0 I}{2\pi R}$ decaying as the inverse distance $R$ from the wire:

![Diagram of magnetic field around a current-carrying wire]

The line integral of $\vec{B}$ around a closed curve $C$ does not have any direct physical interpretation (as does the line integral of $\vec{E}$) but
turns out to be a nice way to characterize the relationship between $\vec{B}$ and its sources—much like Gauss’s law and $\vec{E}$.

Let’s start with a curve $C$ that’s a circle of radius $R$, centered on the wire and in a plane perpendicular to the wire.

\[ \text{d}r^\perp = \text{line element along } C \]

Because our choice of line element is everywhere parallel to $\vec{B}$,
\[ \oint_C \mathbf{B} \cdot d\mathbf{r} = \oint_C \left( \frac{\mu_0 I}{2\pi R} \right) dl = \frac{(\mu_0 I)}{2\pi R} \oint_C dl = \mu_0 I \]

a result that is independent of the radius of \( C \). We will now show that the line integral is independent of almost all properties of \( C \), in the case when \( C \) is not just a simple circle (analogous to the insensitivity, in Gauss's law, of the shape of the surface \( S \)).

So let's imagine a complicated closed curve \( C' \). 

The first thing we can do is project $C$ into a plane perpendicular to the wire. If the resulting curve is $C_{\perp}$, then

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \oint_{C_{\perp}} \mathbf{B} \cdot d\mathbf{r}$$
The line integral is unchanged under this projection because:

(1) \( \vec{B} \) is unchanged, when translated along parallel to the wire:

\[
\vec{B} \cdot d\vec{r} = B_{||} \cdot d\vec{r}_{||} + B_{\perp} \cdot d\vec{r}_{\perp} = B \cdot d\vec{r}_{\perp}
\]

(2) \( \vec{B} \) has no component parallel to the wire:

The second thing we can do is project each line element \( d\vec{r}_{\perp} \)
of \( C_{\perp} \) onto a circle of radius \( R \):

\[
d\vec{r}_{\perp} = d\vec{r} + R'd\phi \hat{\phi}
\]

\[
d\vec{r}_{\perp}' = 0 + R'd\phi \hat{\phi}
\]

(drawing on next page)
\[ \overrightarrow{B} \cdot d\overrightarrow{r}_+ = \overrightarrow{B} \cdot (R'd\phi \hat{\phi}) = \left( \frac{\mu_0 I}{2\pi R} \right) R'd\phi \]

\[ \overrightarrow{B} \cdot d\overrightarrow{r}_- = \left( \frac{\mu_0 I}{2\pi R} \right) R d\phi \]

So we only need to evaluate \( \overrightarrow{B} \) on the circle of radius \( R \) and the line elements \( d\overrightarrow{r}_+ \) all lie on this circle. When all the line elements are projected this way the result
is a circular curve $C'$ which wraps around the wire the same number of times, and in the same sense, as the original curve $C$.

![Diagram of a circular curve $C'$ wrapping around a wire.]

But this is just the very first line integral we performed, or multiple instances of it. For an arbitrary curve $C$ we therefore have

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I \cdot N$$
\[ N = \text{number of times C wraps around wire}. \]

\( N \) can be zero, when \( C \) does not wrap around at all, or negative, when \( C \) wraps around in a left-handed sense. The "sense" of the wrapping is determined in part by the direction of \( I \). Changing the direction of \( I \) changes the sense of the wrapping, and therefore the sign of \( N \). Equivalently, we can change the sign of \( I \) (to reverse its direction) and this also changes the sign of \( \mu_0 I N \).

Since the superposition principle applies \( \to \vec{E} \), it should also apply to \( \vec{B} \). The magnetic field
created by multiple wires carrying currents $I_1, I_2, \ldots$ should be the sum of their individual magnetic fields $\mathbf{B}_1, \mathbf{B}_2, \ldots$. Let $\mathbf{B}$ be this sum (the net magnetic field), then:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \oint_C \mathbf{B}_1 \cdot d\mathbf{r} + \oint_C \mathbf{B}_2 \cdot d\mathbf{r} + \ldots$$

$$= \mu_0 I_1 N_1 + \mu_0 I_2 N_2 + \ldots$$

$$= \mu_0 I_{\text{enc}}$$

The "enclosed" current $I_{\text{enc}}$ depends on both the currents flowing in the wires and the number of times $C$ wraps around each of them.