Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 28: April 14

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28.1 Adiabatic invariance (continued)

28.1.1 Stability of the Levitron® (continued)

In the absence of torques we can model the Levitron as a freely precessing symmetric top. Its angular momentum ("spin") $\mathbf{S}$ will be a fixed vector in space about which the top's symmetry axis $\hat{\mathbf{3}}$ rapidly precesses (lecture 6). There is a simple mechanism that aligns $\hat{\mathbf{3}}$ with $\mathbf{S}$ when the symmetric top is oblate, that is, when I_3 is the largest moment of inertia. First note that the Levitron, like most tops, is a solid of revolution about the $\hat{\mathbf{3}}$ axis and therefore experiences the least aerodynamic drag when $\hat{\mathbf{3}}$ and $\mathbf{S}$ are aligned. When this is not the case, the wobbling top looses energy rapidly to the surrounding air. Since the rotational energy is $S^2/2I$, when $\mathbf{S}$ is parallel to a principal axis, the energy at fixed angular momentum is minimized when we choose the principal axis with largest moment ($I = I_3$). In the rest of the analysis we therefore assume that the spin $\mathbf{S}$, the symmetry axis $\hat{\mathbf{3}}$, and the Levitron's magnetic moment $\boldsymbol{\mu}$ are all parallel.

The magnetic field through which the Levitron moves exerts a torque on the spinning top:

$$\begin{aligned} \dot{\mathbf{S}} &= \boldsymbol{\mu} \times \mathbf{B} && (28.1) \\ &= \left(\frac{\mu B}{S}\right) \mathbf{S} \times \hat{\mathbf{b}}. && (28.2) \end{aligned}$$

In the last line we replaced the vectors in the cross product with parallel vectors and compensated with a scalar prefactor. The symbol $\hat{\mathbf{b}}$ represents the unit vector-field giving the direction of the magnetic field. We also know that the spin magnitude, S, is a constant because the rate-of-change of $\mathbf{S}$, by the cross-product formula, is always perpendicular to itself.

The combination

$$\omega(\mathbf{r}) = \frac{\mu B(\mathbf{r})}{S} \tag{28.3}$$

is the forced precession frequency of the Levitron's symmetry-axis/spin about the local magnetic field axis, $\hat{\mathbf{b}}$. While this is slow compared to the angular velocity S/I of the top, we assume this intermediate scale is still fast relative to the time scale of center-of-mass motion. In other words, we assume conditions are such that there are many precession periods before the top has noticed a change in the direction of $\hat{\mathbf{b}}$.

An analysis similar to what we did for the pendulum, but applied to a top subject to the torques produced by a slowly varying magnetic field, identifies the projection of the spin onto the magnetic field axis as an adiabatic invariant[1]:

$$I = \mathbf{S} \cdot \hat{\mathbf{b}}. \tag{28.4}$$

[1] Try not to confuse the action variable I with the moment of inertia!

This seems plausible, given that $\mathbf{S}$ (with constant magnitude) maintains a constant angle with respect to $\hat{\mathbf{b}}$ while the latter is not changing. By now we are not suprised that I has units of action.

In the Levitron the time dependence of $\hat{\mathbf{b}}$ comes about not through external control but indirectly through the top's motion through the magnetic field. This is probably the most common way that adiabatic invariance is applied more generally. Instead of taking the limit of longer and longer times T over which a parameter is changed externally, we introduce a frequency Ω that characterizes the slowness of the larger system that modifies the conditions of the subsystem (whose periodic motion has frequency ω). In the case of the Levitron, Ω corresponds to the frequency of harmonic motion of the top's position.

Our proof of adiabatic invariance (for oscillators with one degree of freedom like the pendulum) showed that all terms of order $\epsilon^n \propto (\Omega/\omega)^n$ in the Taylor series for changes ΔI in the action exactly vanish. This still allows for non-invariance that behaves as $\Delta I \propto \exp(-c\,\omega/\Omega)$, and at the end of the lecture we use quantum mechanics to make this form plausible. Thanks to the exponential behavior, the difference between exact and near invariance is practically negligible even for modest separations of time scales.

Returning to the question of Levitron stability, we use invariance of I to re-express the magnetic dipole energy of the previous lecture:

$$\begin{aligned} -\boldsymbol{\mu}\cdot\mathbf{B}(\mathbf{r}) &= -\left(\frac{\mu}{S}\mathbf{S}\right)\cdot\hat{\mathbf{b}}(\mathbf{r})B(\mathbf{r}) && (28.5)\\ &= -\left(\frac{\mu I}{S}\right)\cdot B(\mathbf{r}). && (28.6)\end{aligned}$$

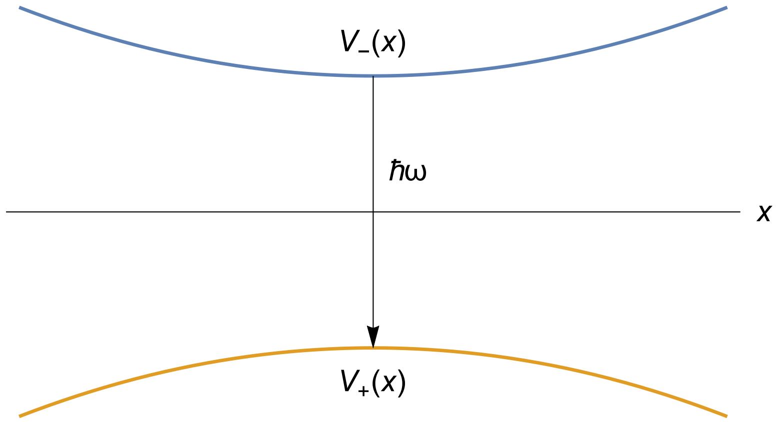
Note that $B(\mathbf{r})$ is the position-dependent *magnitude* of the magnetic field. In our earlier analysis (lecture 27), where the top was not adiabatically precessing but fixed in direction, this potential energy function was replaced by $B_z(\mathbf{r})$. Fortunately, unlike $B_z(\mathbf{r})$, the magnitude function $B(\mathbf{r})$ does not satisfy the Laplace equation and it is in fact possible to construct magnetic fields where $B(\mathbf{r})$ has a local minimum. When we spin the Levitron so that $I < 0$, the resulting potential energy function will be stable in all three dimensions.

28.1.2 Spin-1/2 particle in a magnetic trap

The stabilizing mechanism of the Levitron has a direct counterpart in the magnetic traps used to form cold atomic gases. In place of the spinning top we have an atom with magnetic moment $\boldsymbol{\mu}$ parallel to its angular momentum $\mathbf{S}$, which will have the fixed magnitude $\hbar/2$ when the atom has spin quantum number 1/2. The action variable $I = \mathbf{S}\cdot\hat{\mathbf{b}}$ takes two discrete values, $I = \pm\hbar/2$, for states of definite energy. These energies are

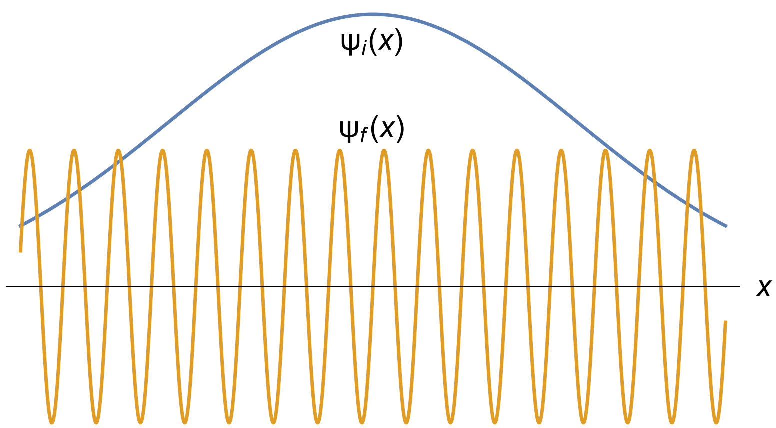
$$V_{\pm}(\mathbf{r}) = \pm\mu B(\mathbf{r}), \tag{28.7}$$

where again the dependence on position is through the magnitude of the magnetic field. To construct an atom trap one only has to engineer a magnetic field where $B(\mathbf{r})$ has a local minimum (for the Levitron one must also take into account the gravitational potential energy). The figure below shows the two potential functions along an axis (x) that runs through the minimum of $B(\mathbf{r})$.



Adiabatic invariance in the context of this quantum particle system is the fact that the system maintains the same I, either $+\hbar/2$ or $-\hbar/2$, and corresponding energy even while its position is changing. The energy level sketch shows that it is actually the upper state that traps the particle: a particle in the lower energy state sees a potential maximum that propels it away from the center of the trap.

We will make a rough calculation in the quantum mechanics formalism to estimate the degree of action non-invariance. Non-invariance is directly manifested by the small rate that trapped atoms are ejected from the trap as a result of transitions from the upper to the lower energy level. Our rough calculation will take proper account of the very different spatial wave functions of the initial and final states, and ignore the relatively slow spatial variation of the perturbation responsible for the transition. Initially the atom wavefunction is a broad Gaussian, $\Psi_i \propto e^{-(Kx)^2}$, appropriate for the (approximately harmonic) trapping potential $V_-(x)$. After it makes the transition to $V_+(x)$ and acquires a large energy (much greater in scale than the trapping potential), its nearly free-particle wavefunction is a rapidly oscillating plane-wave: $\Psi_f \propto e^{-ikx}$.



We can estimate the frequency Ω of the harmonic motion of the atom's position by the kinetic energy in the Gaussian for an atom of mass M:

$$\hbar\Omega \approx \frac{(\hbar K)^2}{2M}. \tag{28.8}$$

When the atom makes a transition to the lower level, somewhere near the center of the trap, its kinetic energy increases by $2\mu B = 2S\omega = \hbar\omega$. Relating this to the final spatial wave function we obtain

$$\hbar\omega \approx \frac{(\hbar k)^2}{2M}. \tag{28.9}$$

Comparing (28.8) and (28.9),

$$\frac{\Omega}{\omega} = \left(\frac{K}{k}\right)^2. \tag{28.10}$$

The most important contribution to the transition probability is the wave function "overlap" integral of the very different initial and final states:

$$\begin{aligned}
\text{transition prob.} \quad &\propto \quad \left|\int \Psi_f^* \Psi_i \, dx\right|^2 && (28.11)\\
&\propto \quad \left|\int e^{-(Kx)^2 + ikx} \, dx\right|^2 && (28.12)\\
&= \quad \left|\int e^{-(Kx - ik/(2K))^2 - k^2/(2K)^2} \, dx\right|^2 && (28.13)\\
&\propto \quad e^{-(1/2)(k/K)^2} && (28.14)\\
&= \quad e^{-(1/2)(\omega/\Omega)}. && (28.15)
\end{aligned}$$

The Taylor series of this function in the variable $\epsilon = \Omega/\omega$ about $\epsilon = 0$ vanishes to all orders, the same result we found in classical mechanics for the degree of action non-invariance.