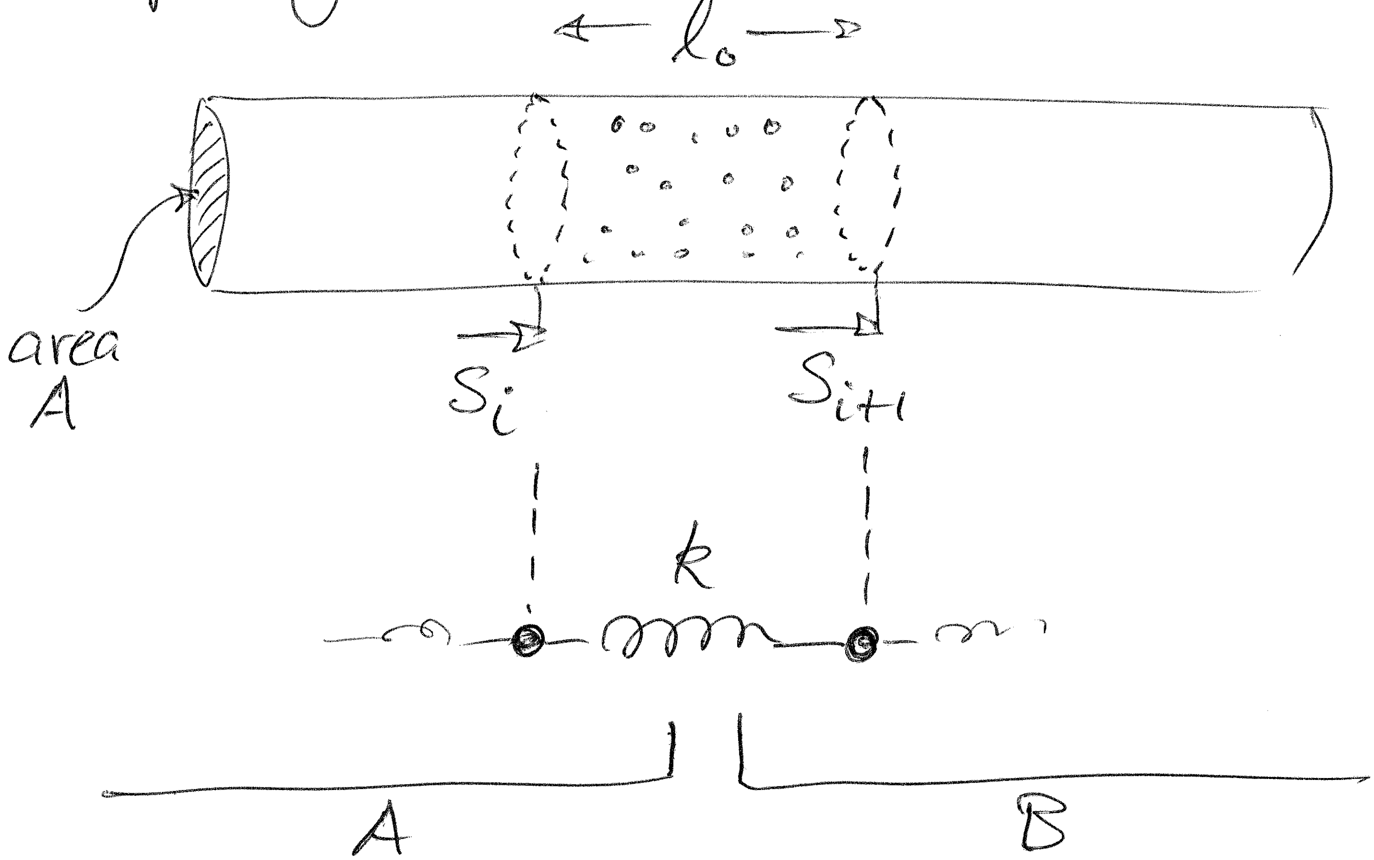
Lecture 28

energy flow in a sound wave

Recall how we represented the wave dynamics of a material (gas, liquid) by a point-mass & spring model:



$$P_{\rightarrow} = \underbrace{k(s_i - s_{i+1})}_{\text{force on mass } i+1} \underbrace{\dot{s}_{i+1}}_{\text{velocity of mass } i+1}$$

$$= -k l_0 \left(\frac{s_{i+1} - s_i}{l_0} \right) \dot{s}_{i+1}$$

$$= -k l_0 \left(\frac{\partial s}{\partial x} \right) \left(\frac{\partial s}{\partial t} \right)$$

In lecture 22 we learned how k is related to a volume derivative of pressure:

$$k = -A^2 \left. \frac{dP}{dV} \right|_{V_0}$$

$V_0 = A l_0$
= equilibrium volume

Substituting this into our equation for $P_{\rightarrow}$ will give something that depends on A — as it should.

Because the medium of sound is three-dimensional (unlike the stretched string), the power also depends on A — it's proportional to it. We therefore define "power per unit area" in the case of sound waves:

$$I = \text{"intensity"} = \frac{P_{\rightarrow}}{A} = \frac{\text{energy}}{\text{time} \times \text{area}}$$

$$= \text{energy "flux"}$$

$$I = -\left(-A l_0 \left.\frac{dP}{dV}\right|_{V_0}\right) \frac{\partial s}{\partial x} \frac{\partial s}{\partial t}$$

$$= -B \frac{\partial s}{\partial x} \frac{\partial s}{\partial t}, \quad B = \text{bulk modulus}$$

We would like to express I in terms of the pressure amplitude Δp of the sound wave. We already know from a previous lecture

$$\Delta p = -B \frac{\partial s}{\partial x} .$$

But I also involves $\frac{\partial s}{\partial t}$.

We can relate the two partial derivatives when we have a <u>running wave</u>:

$$s(x,t) = f(x - v t)$$

For $v > 0$ this wave runs to the right.

$$\frac{\partial s}{\partial x} = f' \quad , \quad \frac{\partial s}{\partial t} = -v f'$$

$$\Rightarrow \quad \frac{\partial s}{\partial t} = -v \frac{\partial s}{\partial x} \quad \leftarrow \text{only true for } \underline{\text{running}} \text{ wave}$$

Return to $\underline{I}$ formula:

$$\underline{I} = -B \frac{\partial s}{\partial x}\left(-v \frac{\partial s}{\partial x}\right)$$

$$= Bv \left(\frac{\partial s}{\partial x}\right)^2$$

$$= Bv \left(-\frac{\Delta p}{B}\right)^2$$

Final result: $\underline{I} = \frac{v}{B} (\Delta p)^2$

This tells us the <u>instantaneous</u>

intensity, since the
pressure amplitude Δp is a function of x and t. For a sinusoidal pressure wave,

$$\Delta p(x,t) = \Delta p_{max} \cos(kx - \omega t),$$

we can work out the time-average of $(\Delta p)^2$:

$$\langle (\Delta p)^2 \rangle = \Delta p_{max}^2 \underbrace{\langle \cos^2(kx - \omega t) \rangle}_{1/2}$$

Hence, for sinusoidal pressure waves

$$\langle I \rangle = \frac{1}{2} \frac{v}{B} \Delta p_{max}^2$$

Intensity of EM waves

In Physics 2217 you learned

$$\vec{I} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad \text{(Poynting's vector)}$$

gives the magnitude of the energy flux of an EM wave as well as its direction.

This looks very different from the power/intensity of the other waves, which had the form:

$$I = (\text{const.}) \begin{pmatrix} \text{time deriv.} \\ \text{of something} \end{pmatrix} \begin{pmatrix} \text{space} \\ \text{deriv.} \end{pmatrix}.$$

But there is a more fundamental description of E & M where the similarity is restored.

In this description both $\vec{E}$ and $\vec{B}$ are derivatives of a vector field $\vec{A}$, called the "vector potential":

$$\vec{E} = -\dot{\vec{A}}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

With this,

$$\vec{I} \propto \begin{pmatrix} \text{time deriv.} \\ \text{of } \vec{A} \end{pmatrix} \begin{pmatrix} \text{space deriv.} \\ \text{of } \vec{A} \end{pmatrix}$$

as in the other waves.

Let's work out $\vec{I}$ for the plane EM wave we considered in lecture 10.

$$E_x = E_z = 0, \quad E_y = E_0 \cos(kx - \omega t)$$

$\omega = c|k|$

By allowing k to be negative we can use one formula — the equation for E_y — ~~to~~ describe both right and left-moving waves.

$B_x = B_y = 0$, Faraday: $\frac{\partial E_y}{\partial x} = -\dot{B}_z$

$$\Rightarrow \dot{B}_z = E_0 k \sin(kx - \omega t)$$

$$\Rightarrow B_z = E_0 \frac{k}{\omega} \cos(kx - \omega t)$$

$$\frac{k}{\omega} = \frac{k}{c|k|} = \frac{\mathrm{sgn}(k)}{c}$$

From E_y and B_z we obtain I_x :

$$I_x = \frac{1}{\mu_0} \frac{\mathrm{sgn}(k)}{c} E_0^2 \cos^2(kx - \omega t)$$

time-average:

$$\langle I_x \rangle = \frac{1}{2} \frac{\mathrm{sgn}(k)}{\mu_0 c} E_0^2$$

So energy flows in the same direction the wave propagates ($\mathrm{sgn}(k)$). Let's now assume $k > 0$ and use $c^2 = \frac{1}{\mu_0 \epsilon_0}$:

$$\langle I_x \rangle = \frac{1}{2} c\, \epsilon_0 E_0^2$$

Note: $c \cdot \epsilon_0 E_0^2 = c \cdot \frac{\text{energy}}{\text{vol.}} = \frac{\text{energy}}{\text{area} \times \text{time}}$