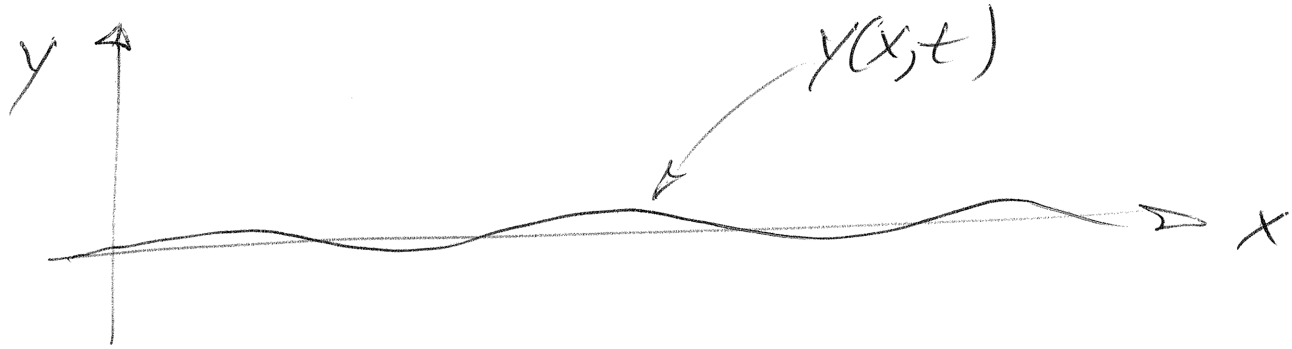
Lecture 27

Energy flow in a wave

Both natural (tsunami) and man-made (laser) waves are capable of propagating large amounts of energy. Let's see how this works in some of the physical systems we've considered in the past.

Transverse waves on a stretched string



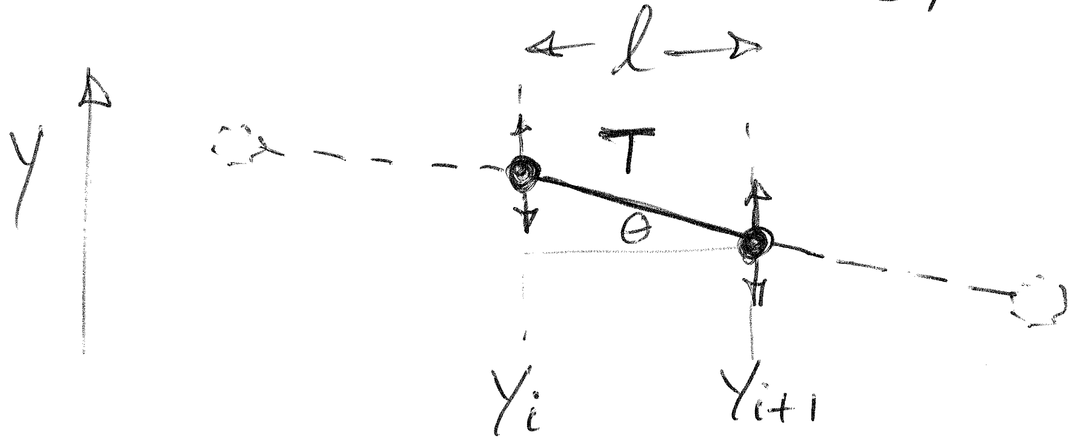
wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

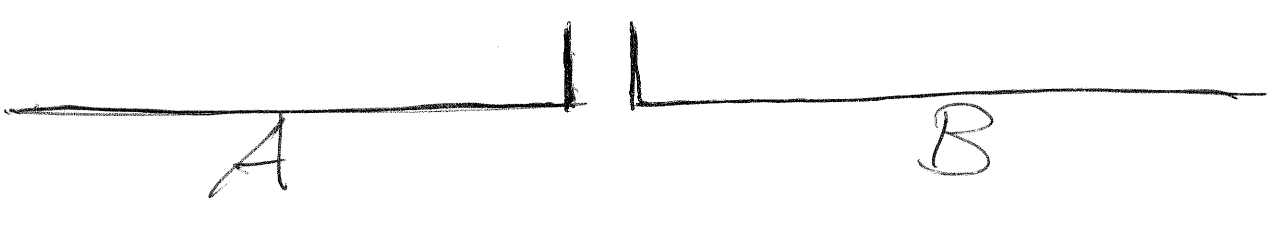
$v = \sqrt{T/\mu}$ $\qquad$ T = tension

$\qquad\qquad\qquad\quad$ μ = mass/length

To work out the flow of energy, separate the string into two parts, A and B, and how the forces between mass-points at their ends transfers energy:



$\theta \ll 1$



$$\begin{pmatrix}\text{rate at which}\\ \text{part A is doing}\\ \text{work on part B}\end{pmatrix} = \begin{pmatrix}\text{rate at which}\\ \text{energy is flowing}\\ \text{from A to B}\end{pmatrix}$$

$$\|$$

$$P_{\rightarrow} = F_y \dot{y}_{i+1}$$

power flow in direction $\rightarrow$ (A to B)

$$= (\sin\theta \; T)\, \dot{y}_{i+1}$$

$$\cong \left(\frac{y_i - y_{i+1}}{l}\right) T \dot{y}_{i+1}$$

$\sin\theta \cong \tan\theta$ for $\theta \ll 1$

$$\cong -T\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$$

Let's evaluate this for a running wave. We will use complex exponentials because it simplifies the calculation of time-averages.

$$y = \mathrm{Re}\left[A e^{ikx-i\omega t}\right]$$

$$= \frac{A}{2} e^{ikx-i\omega t} + \frac{A^*}{2} e^{-ikx+i\omega t}$$

(A^* = complex conjugate of A)

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right) = \left(\frac{ik}{2}\right)\left(A e^{ikx-i\omega t} - A^* e^{-ikx+i\omega t}\right) \times$$

$$\times \left(\frac{-i\omega}{2}\right)\left(A^{ikx-i\omega t} - A^* e^{-ikx+i\omega t}\right)$$

We are interested in the time-averaged power, which we write using angle brackets:

$$\left(\begin{array}{c}\text{time-averaged} \\ \text{power}\end{array}\right) = \langle P_{\rightarrow} \rangle$$

Simple facts:

$$\langle e^{i\omega t} \cdot e^{i\omega t} \rangle = 0$$

$$\langle e^{-i\omega t} \cdot e^{-i\omega t} \rangle = 0$$

$$\langle e^{i\omega t} \cdot e^{-i\omega t} \rangle = \underline{1}$$

Applying these facts to our power expression...

$$\left\langle \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} \right\rangle = \frac{k\omega}{4} \left(-AA^* - A^*A \right)$$

$$AA^* = A^*A = |A|^2$$

$$\langle P_{\rightarrow} \rangle = \frac{1}{2} k\omega |A|^2 \cdot T$$

When $k > 0$, and the wave or wavepacket moves to the right, the averaged power is <u>positive</u>.