Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 27: April 10

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

27.1 Adiabatic invariance (continued)

27.1.1 Proof of invariance

As promised in the previous lecture, we will prove adiabatic invariance of the pendulum action by performing a sequence of time-dependent canonical transformations. We rewrite the time-dependent Hamiltonian for angle-action variables in a form that conveys the generality of the method:

$$\begin{aligned} \mathcal{H}_0(\theta_0, I_0, t) &= \omega I_0 + \epsilon \frac{3}{2} I_0 \left(\frac{1}{l} \frac{dl}{ds} \right) \sin\theta_0 \cos\theta_0 && (27.1) \\ &= \left(\omega(\epsilon t) + \epsilon h_0(\theta_0, \epsilon t) \right) I_0. && (27.2) \end{aligned}$$

We have added the subscript 0 to the Hamiltonian, as well as the angle-action variables, to indicate that these are the first in an infinite sequence of definitions (defined below). Recall that all time dependence is via the dimensionless parameter $s = \epsilon t$, and that time varies between $t = 0$ and $t = 1/\epsilon = T$ in the process.

In the homework assignment you will show that the time-dependent canonical transformation generated by

$$F_0(\theta_0, I_1, t) = \left(\theta_0 - \frac{\epsilon}{\omega(\epsilon t)} \int_0^{\theta_0} h_0(\tilde{\theta}, \epsilon t) d\tilde{\theta} \right) I_1 \tag{27.3}$$

produces the transformed Hamiltonian

$$\mathcal{H}_1(\theta_1, I_1, t) = \left(\omega(\epsilon t) + \epsilon^2 h_1(\theta_1, \epsilon t) \right) I_1, \tag{27.4}$$

which differs from (27.2) by having a different angle-dependent function h_1, and more significantly, an extra power of ϵ multiplying this function. Since the form of the Hamiltonian is unchanged, we can continue applying canonical transformations of this kind (but with angular functions h_1, h_2, etc.) and produce Hamiltonians[1]

$$\mathcal{H}_n(\theta_n, I_n, t) = \left(\omega(\epsilon t) + \epsilon^{n+1} h_n(\theta_n, \epsilon t) \right) I_n, \tag{27.5}$$

for arbitrarily large n.

It would seem that adiabatic invariance now follows from Hamilton's equations applied to $\mathcal{H}_n$,

$$\dot{I}_n = -\frac{\partial \mathcal{H}_n}{\partial \theta_n} = -\epsilon^{n+1} \frac{\partial h_n}{\partial \theta_n} I_n, \tag{27.6}$$

[1] Hamiltonia?

because then

$$I_n(T) = I_n(0) + T \cdot O(\epsilon^{n+1}) = I_n(0) + O(\epsilon^n). \tag{27.7}$$

However, what we really wanted to prove was adiabatic invariance of the original action variable I_0, not the multiply transformed variable I_n. To fix this detail we revisit the transformation formula:

$$I_n = \frac{\partial F_n}{\partial \theta_n} = \left(1 - \frac{\epsilon^{n+1}}{\omega} h_n(\theta_n, \epsilon t)\right) I_{n+1}. \tag{27.8}$$

First consider $n = 0$. From (27.1) we see that $h_0 \propto dl/ds$ would vanish at the endpoints of the process ($s = 0$ and $s = 1$) if we arranged for the string-length derivative to vanish there. This property of an adiabatic process, that there should be no discontinuous derivatives when the process is initiated and terminated, should be just as important as the slowness ($\epsilon \to 0$) of the process. Assuming h_n vanishes at both endpoints, the transformation formula (27.8) implies

$$s = 0 \quad : \quad I_n(0) = I_{n+1}(0) \tag{27.9}$$

$$s = 1 \quad : \quad I_n(T) = I_{n+1}(T). \tag{27.10}$$

So far we have only arranged for this to be true for $n = 0$. The rule for transforming the Hamiltonian (assigned as homework) shows that h_{n+1} is constructed from h_n and $\partial h_n/\partial s$. Therefore, all the functions h_n would vanish at $s = 0$ and $s = 1$ if not only h_0, but *all* of its s-derivatives vanish at the endpoints of the process. This is not as restrictive a condition as it might seem. For example (verification assigned as homework), a string-length parameterization with behavior of the form

$$l(s) \underset{s \to 0}{\sim} l_1 + f(s) e^{-c/s} \tag{27.11}$$

for $c > 0$ and any rational function $f(s)$ will have vanishing derivatives at all orders. Choosing our function $l(s)$ to have this infinitely smooth behavior at onset and also when the process terminates ($s \to 1$) will ensure that all the functions h_n vanish at the endpoints. By the transformation rule (27.8) all the transformed action variables will therefore agree in value at the endpoints.

The proof of invariance — to arbitrary order in the slowness parameter ϵ — now follows by the following steps:

$$I_0(T) = I_n(T) = I_n(0) + O(\epsilon^n) = I_0(0) + O(\epsilon^n). \tag{27.12}$$

Question: Suppose the dimensionless product $\epsilon\, T_0$ is sufficiently small, where T_0 is the period of the motion. Does the result we just proved imply I_0 is exactly conserved?

27.1.2 Adiabatic invariance and the stability of the Levitron®

The parameter — under whose variation some action variable I is adiabatically invariant — is in many instances itself a dynamical variable of a larger system and only appears to be "external". A nice example of this form of adiabatic invariance is the levitating magnetic top toy: the Levitron.

The Levitron is a scheme for levitating a small bar magnet by placing it over the fixed magnetic field of an opposing bar magnet. The levitating magnet resists torques that threaten to flip it over by rapidly spinning as a top about its magnetization axis. While this much of the physics is certainly relevant for a levitating equilibrium, the *stability* of the equilibrium relies crucially on very different physics: adiabatic invariance.

Suppose the angular momentum, or "spin" $\mathbf{S}$ of the top is so great that its magnetic moment $\boldsymbol{\mu} \propto \mathbf{S}$ maintains a fixed orientation in space (the large torque required to change its axis is not present in its environment).

Let the fixed axis of the top coincide with the vertical axis of gravity and define the z-axis. The potential energy of the top now only depends on its position through the following potential energy function:

$$V(\mathbf{r}) = Mgz - \boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{r}) \tag{27.13}$$
$$= Mgz - \mu B_z(\mathbf{r}). \tag{27.14}$$

The static magnetic field $\mathbf{B}$ of the fixed permanent magnet can be expressed as the gradient of a potential Φ that satisfies the Laplace equation outside the source:

$$\mathbf{B} = \nabla\Phi, \qquad \nabla^2\Phi = 0. \tag{27.15}$$

These equations imply

$$B_z = \frac{\partial\Phi}{\partial z}, \qquad \nabla^2 B_z = \frac{\partial}{\partial z}\left(\nabla^2\Phi\right) = 0. \tag{27.16}$$

Since the gravitational energy Mgz also satisfied the Laplace equation, so does the entire potential energy function V. But $\nabla^2 V$ is the trace of the Hessian matrix of V, and a value of zero implies that not all three eigenvalues can be positive. The net potential energy function will therefore always have at least one unstable direction![2]

The possibility of a stable levitating state is only resolved by taking into consideration motion at three very different time scales:

fast:	spinning of top about its axis
intermediate:	precession of the spin axis
slow:	center-of-mass motion.

The second of these — precession — is the key to understanding the Levitron's stability and is the subject of the next lecture.

[2]The electrostatic analog of this — non-existence of static equilibrium among electric charges — is known as Earnshaw's theorem.