Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 26: March 31

Instructor: Veit Elser

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26.1 Adiabatic invariance

26.1.1 Invariance in a slow process

When we encountered invariance properties in earlier lectures, in particular in connection with Noether's theorem (lecture 17), the invariance or conservation law was a direct consequence of a continuous symmetry. In this lecture we introduce a general kind of invariance that is unrelated to symmetries of the system, and instead appears in the context of external parameters that are slowly varied. We do not call this form of invariance a "conservation law" because the constancy is not perfect. On the other hand, the degree of quasi-conservation is improved exponentially with slowness, and for even modest parameter rates-of-change the invariance is practically indistinguishable from a true conservation law.

The new kind of invariance only applies to periodic motion, and when the parameters of the Hamiltonian are changed only slightly in each period. The degree of quasi-conservation depends on the ratio of two time scales: the period of the near-periodic motion T_0, and the long time period T over which the parameters are varied. One of our goals in the next two lectures is to show that the invariance is better than $(T_0/T)^n$, for arbitrarily large exponents n.

The property that is quasi-conserved is the action I enclosed by the periodic orbit in phase space, and the improved constancy with slowness of the external parameter variation is called "adiabatic invariance". In previous lectures we encountered links between the action in mechanics and concepts in quantum mechanics. This instance of action is no exception. Adiabatic invariance is easier to understand in the quantum context. For example, we analyze a simple quantum system in lecture 28 to argue that the inexactness of adiabatic invariance decays with the time scales as $\exp(-c\,T/T_0)$, consistent with the result we will obtain in classical mechanics.

26.1.2 Adiabatic variation of the pendulum

Consider the l (string length) dependence of our pendulum Hamiltonian:

$$\mathcal{H}(q,p,t) = \frac{p^2}{2Ml^2(t)} + \frac{1}{2}Mg\,l(t)q^2. \tag{26.1}$$

The contours of $\mathcal{H}$ are ellipses with major/minor axes that depend on l.

Exercise: Sketch contours of $\mathcal{H}$ in two phase-spaces side by side, one with $l = l_1$ and the other with $l = l_2$, where $l_1 > l_2$.

Suppose at time $t = 0$ we have $l(0) = l_1$ and the pendulum orbit is one of the ellipses in the $l = l_1$ phase space. Over a long period T we then slowly decrease the length to $l(T) = l_2$, and the pendulum will then be moving on one of the ellipses in the $l = l_2$ phase space. Can we predict the final orbit given the initial orbit?

There is no reason for the pendulum to have the same energy at the two values of l, since the energy need not be conserved when the Hamiltonian is time-dependent. As mentioned above, instead it is the action that is (quasi)-conserved. Before we try to prove this we will examine the consequences of this basic fact.

Suppose q_0 is the maximum amplitude (for $p = 0$), and p_0 is the maximum momentum (for $q = 0$), for some string length l. We then have the following relationship:

$$\frac{p_0^2}{2Ml^2} = \frac{1}{2}Mg\,lq_0^2. \tag{26.2}$$

The action for this periodic motion is

$$I = \frac{1}{2\pi}(\text{area of ellipse with major/minor axes } q_0 \text{ and } p_0) = \frac{1}{2}q_0 p_0. \tag{26.3}$$

Keeping track of only the l dependence we can make the following sequence of statements:

$$I = \text{const.} \quad \Rightarrow \quad q_0 p_0 = \text{const.} \quad \Rightarrow \quad q_0(l^{3/2}q_0) = \text{const.} \quad \Rightarrow \quad q_0 \propto \frac{1}{l^{3/4}}. \tag{26.4}$$

This shows how the amplitude increases as the string is shortened. Similarly, we can obtain the l dependence of the energy:

$$\mathcal{H} \propto lq_0^2 \propto \frac{1}{\sqrt{l}}. \tag{26.5}$$

A more direct way to obtain the l dependence of the energy is to use angle-action variables,

$$\mathcal{H}' = \omega I = \sqrt{g/l}\, I \propto \frac{1}{\sqrt{l}} \tag{26.6}$$

since the action I is constant in an adiabatic process.

It's possible to check these scaling results for $q_0(l)$ and $\mathcal{H}(l)$ using elementary Newtonian mechanics — with no reference to action. A convenient approach (assigned as homework) is to implement string-shortening by a collar through which the string is threaded and which moves slowly downward with a speed v. By evaluating the power input to the pendulum by the external agent that is moving the collar in the limit of small v, one can infer the pendulum energy $\mathcal{H}(l)$ and indirectly $q_0(l)$. Whereas the Newtonian method works, the results obtained thereby seem mysterious and do not illuminate a general principle.

26.1.3 Order-by-order adiabatic invariance using generating functions

To demonstrate adiabatic invariance of the action for the pendulum in a way that generalizes to other systems we use angle-action variables. We have already seen the transformation to these variables in lecture 25; what remains is to add the correction that arises when the transformation is time dependent.

We introduce time dependence in the following way. The string length l is parameterized by a dimensionless parameter s that varies between 0 and 1, such that $l(0) = l_1$ and $l(1) = l_2$. By setting $s = \epsilon\, t$, where t is the time, the adiabatic limit is realized by taking the limit $\epsilon \to 0$ (time $1/\epsilon$ is required to make the change).

Here is the generating function from lecture 25 we used to transform the pendulum Hamiltonian to angle-action variables:

$$F(q, \theta, t) = \frac{1}{2} M \sqrt{g}\, l^{3/2}(\epsilon t)\, q^2 \cot\theta. \tag{26.7}$$

The correction to $\mathcal{H}' = \omega I$ that comes about from the time dependence of the transformation is

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{3}{4} M \sqrt{g}\, \epsilon \sqrt{l} \frac{dl}{ds}\, q^2 \cot\theta & (26.8) \\ &= \frac{3}{4} M \omega\, \epsilon\, l \frac{dl}{ds}\, q^2 \cot\theta. & (26.9) \end{aligned}$$

We need to express this in terms of the transformed variables, in particular (lecture 25),

$$q^2 \cot\theta = \frac{2I}{M\omega l^2} \sin\theta \cos\theta. \tag{26.10}$$

The transformed Hamiltonian is therefore

$$\mathcal{H}'(\theta, I, t) = \omega I + \epsilon \frac{3}{2} I \left(\frac{1}{l} \frac{dl}{ds} \right) \sin\theta \cos\theta. \tag{26.11}$$

As a result of the new (second) term, the action *variable* is no longer time independent:

$$\dot{I} = -\frac{\partial H'}{\partial \theta} = -\epsilon \frac{3}{2} I \left(\frac{1}{l} \frac{dl}{ds} \right) \cos 2\theta. \tag{26.12}$$

It appears that the invariance of the action I is only good to order ϵ. However, the question we asked at the beginning of the lecture concerned the action I defined by the *phase space area enclosed over a complete period*. This we will see is invariant to arbitrary powers of the small parameter ϵ[1]. We do this in the next lecture by performing a sequence of time-dependent canonical transformations that order-by-order improve the agreement between the two definitions of I (variable vs. enclosed area).

[1] You can see how that might work from the fact that $\cos 2\theta$ oscillates in sign over the course of one period.