25.1 Angle action variables

There is a special choice of variables we use for periodic motion. Since the motion always stays on a “surface” of constant $H$, and for one degree of freedom these surfaces are contours in a two dimensional phase space, the constant $H$ contours will be closed curves when the motion is periodic. The “action variable”, traditionally written $I$, is defined as

$$2\pi I(q,p) = \text{phase space area (action) enclosed by } H \text{ contour passing through } (q,p)$$

This variable plays the role of the transformed momentum variable, denoted $P$ in previous lectures. By construction, $I$ is always non-negative and the Hamiltonian (for the new variables) is solely a function of $I$:

$$H' = h(I).$$

The transformed coordinate variable, that $I$ is conjugate to (and previously denoted $Q$), is written $\theta$ and called the “angle variable”. We can define $\theta$ by a geometrical construction as follows. Draw a curve, starting at the point $(q_0,p_0)$ in phase space where $I = 0$ (the center of all the contours), and crossing all the contours exactly once. Unless the contours have complicated shapes, a straight ray serves this purpose. This ray/curve defines $\theta = 0$ everywhere in phase space. To define $\theta$ elsewhere, we use the phase-space area constancy property. Consider the annular region in the $(q,p)$ plane bounded by contours with action $I$ and $I + \Delta I$. The canonical transformation maps this annulus to a rectangular region in the $(\theta,I)$ plane bounded by $I, I + \Delta I, \theta = 0$ and $\theta = 2\pi$. In order that equal areas of the annulus get mapped to equal areas in the rectangle, we define the angle variable as follows:

$$\theta(q,p)/2\pi = \text{fraction of the annulus area, measured clockwise from } \theta = 0.$$
Exercise: Sketch the \((q,p)\) and \((\theta,I)\) phase spaces side by side, showing some corresponding contours of constant \(I\) (and constant \(H\)). Locate points in the \((q,p)\) plane that correspond to \(\theta = 0\) and \(\theta = \pi/2\).

Hamilton’s equations in the transformed variables are exceedingly simple:

\[
\dot{\theta} = + \frac{\partial H'}{\partial I} = \frac{dh}{dI}, \quad \quad \quad \quad \quad (25.4)
\]
\[
\dot{I} = - \frac{\partial H'}{\partial \theta} = 0. \quad \quad \quad \quad \quad (25.5)
\]

The second equation implies \(I = I_0\) is constant — something we already knew since \(H' = h(I)\) will only be constant if \(I\) is constant. The solution of the first equation is almost as simple:

\[
\dot{\theta} = \frac{dh}{dI} \bigg|_{I = I_0}^{} = \omega_0 = \text{constant} \quad \quad \quad \quad \quad (25.6)
\]
\[
\theta(t) = \omega_0 t + \theta_0. \quad \quad \quad \quad \quad (25.7)
\]

The periodicity of orbits is built into the topology of the \((\theta,I)\) phase space, as we declare the points \((0,I)\) and \((2\pi,I)\) equivalent. In other words, the two sides of the semi-infinite rectangle are sewn together, topologically, into a cylinder.

A useful way to think about the transformed Hamiltonian (25.2) is in terms of its derivative. Let \(T_0\) be the period of the motion when the action is \(I_0\). By (25.7) we have \(\omega_0 T_0 = 2\pi\), and therefore, using (25.6),

\[
\frac{dh}{dI} \bigg|_{I = I_0} = \frac{2\pi}{T_0}. \quad \quad \quad \quad \quad (25.8)
\]

25.1.1 Transforming the simple harmonic oscillator to angle action variables

Let’s construct the actual transformation to angle-action variables for the simplest model of periodic motion: the harmonic oscillator. We will consider a pendulum where the length of the string \(l\) may be varied over time. Since the canonical transformation in this case is time-dependent, we will eventually need the full machinery of generating functions (lecture 24) to work out the transformed Hamiltonian \(H'\).

Assuming a massless string supporting a point mass \(M\), the Lagrangian of the system is

\[
\mathcal{L} = T - V = \frac{1}{2} M \left( l \dot{\phi} \right)^2 - Mgl(1 - \cos \phi) \quad \quad \quad \quad \quad (25.9)
\]
\[
\approx \frac{1}{2} M \left( l \dot{\phi} \right)^2 - \frac{1}{2} Mgl \phi^2. \quad \quad \quad \quad \quad (25.10)
\]

where \(\phi\) is the angle from the vertical, and we are interested only in small amplitude oscillations. To preserve continuity with the earlier notation, we make the replacement \(\phi \rightarrow q\) for the coordinate and define
the conjugate momentum in the usual way:

\[ p = \frac{\partial L}{\partial \dot{q}} = Ml^2 \dot{\theta}. \]  

(25.11)

We write the resulting Hamiltonian in two ways,

\[ H = p\dot{q} - L = \left( \frac{1}{2Ml^2} \right) p^2 + \left( \frac{Mgl}{2} \right) q^2 \]

(25.12)

\[ = \left( \frac{1}{2Ml^2} \right) p^2 + \left( \frac{M\omega^2 l^2}{2} \right) q^2, \]

(25.13)

where

\[ \omega = \sqrt{g/l}. \]  

(25.14)

When we later study the effect of a time dependent length \( l \) we will have to remember that this makes the frequency \( \omega \) also time dependent.

Here is the generating function that will transform the \((q,p)\) phase space of the harmonic oscillator to the angle action \((\theta,I)\) phase space:

\[ F(q,\theta,t) = \frac{1}{2} M\omega l^2 q^2 \cot \theta. \]  

(25.15)

Note that the second argument of \( F \) is occupied by \( \theta \), our name for the transformed coordinate variable. The canonical transformation is determined by the following two equations:

\[ p = \frac{\partial F}{\partial q} = M\omega^2 l^2 q \cot \theta \]

(25.16)

\[ I = -\frac{\partial F}{\partial \theta} = -\frac{1}{2} M\omega^2 l^2 q^2 \left( \frac{-1}{\sin^2 \theta} \right). \]  

(25.17)

Solving for \( q \) in the second equation,

\[ q = \sqrt{\frac{2I}{M\omega^2 l^2}} \sin \theta, \]

(25.18)

we see that the “angle variable” \( \theta \) is not the same as the angle \( q = \phi \) of the pendulum. Substituting this \( q \) into the first equation, we obtain

\[ p = \sqrt{2IM\omega^2 l^2} \cos \theta. \]  

(25.19)

Finally, substituting these formulas expressing \((q,p)\) in terms of \((\theta,I)\) into the Hamiltonian (25.13), we obtain

\[ \mathcal{H}'(\theta,I,t) = \omega I + \frac{\partial F}{\partial t}. \]  

(25.20)

When the transformation is time-independent \((\partial F/\partial t = 0)\) the transformed Hamiltonian is especially simple: a linear function of \( I \). This should not be surprising because we know that the period (25.8) of the simple harmonic oscillator is a constant, independent of amplitude (enclosed action).