Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 25: March 29

Instructor: Veit Elser

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25.1 Angle action variables

There is a special choice of variables we use for *periodic motion.* Since the motion always stays on a "surface" of constant $\mathcal{H}$, and for one degree of freedom these surfaces are contours in a two dimensional phase space, the constant $\mathcal{H}$ contours will be closed curves when the motion is periodic. The "action variable", traditionally written I, is defined as

$$2\pi I(q,p) = \text{phase space area (action) enclosed by } \mathcal{H} \text{ contour passing through } (q,p) \tag{25.1}$$

This variable plays the role of the transformed momentum variable, denoted P in previous lectures. By construction, I is always non-negative and the Hamiltonian (for the new variables) is solely a function of I:

$$\mathcal{H}' = h(I). \tag{25.2}$$

The transformed coordinate variable, that I is conjugate to (and previously denoted Q), is written θ and called the "angle variable". We can define θ by a geometrical construction as follows. Draw a curve, starting at the point (q_0, p_0) in phase space where $I = 0$ (the center of all the contours), and crossing all the contours exactly once. Unless the contours have complicated shapes, a straight ray serves this purpose. This ray/curve defines $\theta = 0$ everywhere in phase space. To define θ elsewhere, we use the phase-space area constancy property. Consider the annular region in the (q,p) plane bounded by contours with action I and $I + \Delta I$. The canonical transformation maps this annulus to a rectangular region in the (θ, I) plane bounded by I, $I + \Delta I$, $\theta = 0$ and $\theta = 2\pi$. In order that equal areas of the annulus get mapped to equal areas in the rectangle, we define the angle variable as follows:

$$\theta(q,p)/2\pi = \text{fraction of the annulus area, measured clockwise from } \theta = 0. \tag{25.3}$$

Exercise: Sketch the (q,p) and (θ,I) phase spaces side by side, showing some corresponding contours of constant I (and constant $\mathcal{H}$). Locate points in the (q,p) plane that correspond to $\theta=0$ and $\theta=\pi/2$.

Hamilton's equations in the transformed variables are exceedingly simple:

$$\dot{\theta} = +\frac{\partial \mathcal{H}'}{\partial I} = \frac{dh}{dI} \tag{25.4}$$

$$\dot{I} = -\frac{\partial \mathcal{H}'}{\partial \theta} = 0. \tag{25.5}$$

The second equation implies $I=I_0$ is constant — something we already new since $\mathcal{H}'=h(I)$ will only be constant if I is constant. The solution of the first equation is almost as simple:

$$\dot{\theta} = \left.\frac{dh}{dI}\right|_{I=I_0} = \omega_0 = \text{constant} \tag{25.6}$$

$$\theta(t) = \omega_0 t + \theta_0. \tag{25.7}$$

The periodicity of orbits is built into the topology of the (θ,I) phase space, as we declare the points $(0,I)$ and $(2\pi,I)$ equivalent. In other words, the two sides of the semi-infinite rectangle are sewn together, topologically, into a cylinder.

A useful way to think about the transformed Hamiltonian (25.2) is in terms of its derivative. Let T_0 be the period of the motion when the action is I_0. By (25.7) we have $\omega_0 T_0 = 2\pi$, and therefore, using (25.6),

$$\left.\frac{dh}{dI}\right|_{I=I_0} = \frac{2\pi}{T_0}. \tag{25.8}$$

25.1.1 Transforming the simple harmonic oscillator to angle action variables

Let's construct the actual transformation to angle-action variables for the simplest model of periodic motion: the harmonic oscillator. We will consider a pendulum where the length of the string l may be varied over time. Since the canonical transformation in this case is time-dependent, we will eventually need the full machinery of generating functions (lecture 24) to work out the transformed Hamiltonian $\mathcal{H}'$.

Assuming a massless string supporting a point mass M, the Lagrangian of the system is

$$\mathcal{L} = T - V = \frac{1}{2}M\left(l\dot{\phi}\right)^2 - Mgl(1-\cos\phi) \tag{25.9}$$

$$\approx \frac{1}{2}M\left(l\dot{\phi}\right)^2 - \frac{1}{2}Mgl\phi^2, \tag{25.10}$$

where ϕ is the angle from the vertical, and we are interested only in small amplitude oscillations. To preserve continuity with the earlier notation, we make the replacement $\phi \to q$ for the coordinate and define the conjugate momentum in the usual way:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = Ml^2 \dot{q}. \tag{25.11}$$

We write the resulting Hamiltonian in two ways,

$$\begin{aligned} \mathcal{H} = p\dot{q} - \mathcal{L} &= \left(\frac{1}{2Ml^2}\right) p^2 + \left(\frac{Mgl}{2}\right) q^2 && (25.12) \\ &= \left(\frac{1}{2Ml^2}\right) p^2 + \left(\frac{M\omega^2 l^2}{2}\right) q^2, && (25.13) \end{aligned}$$

where

$$\omega = \sqrt{g/l}. \tag{25.14}$$

When we later study the effect of a time dependent length l we will have to remember that this makes the frequency ω also time dependent.

Here is the generating function that will transform the (q, p) phase space of the harmonic oscillator to the angle action (θ, I) phase space:

$$F(q, \theta, t) = \frac{1}{2} M\omega l^2 q^2 \cot\theta. \tag{25.15}$$

Note that the second argument of F is occupied by θ, our name for the transformed coordinate variable. The canonical transformation is determined by the following two equations:

$$\begin{aligned} p &= +\frac{\partial F}{\partial q} = M\omega l^2 q \cot\theta && (25.16) \\ I &= -\frac{\partial F}{\partial \theta} = -\frac{1}{2} M\omega l^2 q^2 \left(\frac{-1}{\sin^2\theta}\right). && (25.17) \end{aligned}$$

Solving for q in the second equation,

$$q = \sqrt{\frac{2I}{M\omega l^2}} \sin\theta, \tag{25.18}$$

we see that the "angle variable" θ is not the same as the angle $q = \phi$ of the pendulum. Substituting this q into the first equation, we obtain

$$p = \sqrt{2IM\omega l^2} \cos\theta. \tag{25.19}$$

Finally, substituting these formulas expressing (q, p) in terms of (θ, I) into the Hamiltonian (25.13), we obtain

$$\mathcal{H}'(\theta, I, t) = \omega I + \frac{\partial F}{\partial t}. \tag{25.20}$$

When the transformation is time-independent ($\partial F/\partial t = 0$) the transformed Hamiltonian is especially simple: a linear function of I. This should not be surprising because we know that the period (25.8) of the simple harmonic oscillator is a constant, independent of amplitude (enclosed action).